Against Set Theory

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Since the rise of modern science in the seventeenth century, philosophy has suffered from a need for legitimation. If it is science that provides us with our advanced knowledge of the world, what role is there left for philosophy? Several answers have been given. One answer is that of Wittgenstein. Philosophy is not in the knowledge business at all. In a world in which no one misused language, there would be no need for the restorative activity called 'philosophy'. Philosophy, like medicine, is a response to imperfection. Among the positive answers are three. One assimilates philosophy to natural science and denies there is any sharp distinction. We find this among empiricists such as Mill, Quine and Armstrong. Another attempts to discern some special subject matter and/or method for philosophy, inaccessible to and prior to science. Kant's critical philosophy, Husserl's phenomenology and some strains of linguistic-analytic philosophy are like this. Finally there are those who would assimilate philosophy to the formal sciences of mathematics and logic. This was most popular among seventeenth century rationalists who attempted to do philosophy *more geometrico* but has its modern echoes in philosophy in the phenomenon we may call 'math envy'. Applying set theory promises the philosopher partial relief from math envy, because it can be used to convince sceptics that philosophy too can be hard science. This paper is about why philosophers should stand up on their own and overcome their besottedness with sets.

Set theory in mathematics

Set theory was created single-handedly by Georg Cantor as recently as 130 years ago. Prompted in part by suggestive ideas of Bolzano and adopting Bolzano's term 'Menge', it served as a vehicle for exploring the transfinite. In the 1870s and 1880s Cantor established the use of one-one correspondence as indicating the size or cardinality of a collection of objects, and it was these collections that he termed 'sets'. Here is Cantor's famous 1895 characterization of what a set is:

> Unter einer "Menge" verstehen wir jede Zusammenfassung *M* von bestimmten wohlunterschiedenen Objekten *m* unserer Anschauung oder unseres Denkens (welche die "Elemente" von *M* genannt werden) zu einem Ganzen.[1]

Among the main early achievements of Cantor's set theory were the demonstration that there are more real numbers than natural, rational or algebraic numbers, the demonstration that continua of any dimension are equinumerous with continua of one dimension, and the use of diagonalization to generate an unending sequence of infinite cardinalities. From here he developed the concepts of order type and ordinal number. Set theory soon ran into problems of paradox however, first Burali-Forti's paradox of the greatest ordinal, possibly discovered by Cantor as early as 1895, Cantor's own paradox of the greatest cardinal, discovered by him in 1899, and the Russell-Zermelo paradox of 1901. Nor was Cantor's theory or its subsequent development by Zermelo and others able to answer the continuum problem, namely whether the cardinality of the continuum is or is not the next greater cardinality than the cardinality of the natural numbers, and the results of Gödel and Cohen showed that the set theory developed to date was incapable of giving an answer either way.

Although Cantor himself was not much bothered by the paradoxes, others such as Frege and Russell took them seriously as threatening their attempts to provide a logical foundation for mathematics. Modern mathematical logic in all its complexity emerged from the efforts to salvage as much as possible from the wreckage, and this logic was soon in employment by Russell and others as a vehicle for tackling philosophical as well as logical and mathematical problems. The prestige quite rightfully earned by mathematical logic among philosophers and logicians from Frege to Turing, and its dutiful application to philosophy by Russell, Whitehead, Carnap, Reichenbach, Quine

1. "By a 'set' we understand any collection into a whole *M* of definite and well distinguished objects *m* of our intuition or our thought, which are called the 'elements' of *M*."

and others, help to explain why set theory has acquired an almost unassailable status among philosophers as "hard" theory, to be respected and used but not questioned.

Set theory found application for example in the twentieth-century developments of measure theory (including probability theory) and point-set topology, and by the 1960s it was commonplace in textbooks and articles to present other branches of mathematics using set theory as a general framework. Even as mathematics, set theory, despite this common use of some of its terminology as a *lingua franca*, is a not especially natural, fruitful or useful part of the subject. Traditional number theory, geometry, analysis, function theory, and all mathematics up until Cantor were developed in happy ignorance of set theory. Contrast this historical fact with this takeover bid by a leading modern set theorist:

> All branches of mathematics are developed, consciously *or unconsciously*, in set theory or in some part of it.[2] (Levy 1979, 1f.; my italics)

In his 1911 *An Introduction to Mathematics*, the mathematical counterpart in its classic clarity and accessibility to Russell's *Problems of Philosophy*, Whitehead was easily able to give a balanced and attractive introduction to the subject without once mentioning sets. Modern logicians of the highest calibre, such as Church and Curry, present their work set-free. Most mathematicians are profoundly uninterested in foundational issues. Among those that are interested, a majority choose to present their views in terms of some form of set theory (which of the several set theories available they use varies). They prefer the simplicity and convenience of sets to the notational and conceptual complexities of type theory, though as Frege, Russell, Lesniewski, Church and others recognized, philosophically, types are in some ways more natural than sets. Some mathematicians and philosophers have claimed that category theory makes a better and more mathematically suitable or powerful foundation than set theory.

Even within mathematics some uses of set theory are questionable. It is usual to

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2. So Euclid, Archimedes, Ptolemy, Al Khwarizmi, Newton, Leibniz, Euler, Gauß, Cauchy and many others were all doing unconscious set theory.

interpret numbers of various kinds as sets, but as Benacerraf (1965) showed, this leads to the pseudo-question *which* sets the numbers should be. Numbers are old and very useful, whereas sets are new and problematic. To explicate numbers as sets is to explain the clear by the obscure. The obscurity turns on the fact that set theory has no natural interpretation. Despite Lewis's (1991) heroic attempt to tame set theory via mereology, he was left with the mystery of the singleton: what distinguishes a from $\{a\}$? Singletons enable set theory to do what upset Goodman, to build many things out of the same materials, so one thing can occur more than once in a set. For example in $\{\{a,b\},\{a,c\}\}$ a occurs twice. In Lewis's regimentation this set comes out as $\{\{\{a\} \cup \{b\}\} \cup \{\{a\} \cup \{c\}\}\}$. Without singletons it would just collapse into $\{a,b,c\}$. Both singletons and the empty set stuck like a fishbone in the craw of the twentieth century logician most vehemently opposed to set theory, Stanislaw Lesniewski.

False idols in philosophy

Leaving the status of set theory within mathematics now aside, let us consider its use, that is to say, its misuse, within and around philosophy. Philosophers and others have had the wool pulled over their eyes by authorities in set theory, who try to present sets as something wholly natural and uncontroversial. The technique, usually applied on or about page 1 of a textbook of set theory, is to claim that we are already familiar with sets under some other names or guises, and then trade on this supposed familiarity to sell us a bill of fare which is ontologically far from neutral and far from benign. Here are some authorities:[3]

> Consider a collection of concrete objects, for instance of the apples, oranges etc. in a fruit shop. We may call it a set of fruit, the individual apples etc. being the members (or elements) of the set. Conceiving the collection as a new single concept is an elementary intellectual act.[4] (Fraenkel 1953, 4)

3. I am not being ironical: these are from some of the best books on set theory.
4. So one can buy a set from a fruiterer or supermarket.

> a pack of wolves, a bunch of grapes, or a flock of pigeons are all examples of sets of things.[5] (Halmos 1960, 1)

> In our examples, sets consisted of concrete and familiar objects, but once we have sets, we can form sets of sets, e.g. the set of all football teams.[6] (Van Dalen, Doets and de Swart 1978, 1)

> Intuitively speaking, a *set* is a definite collection, a plurality of objects of any kind, which is itself apprehended as a single object. For example, think of a lot of sheep grazing in a field. They are a collection of sheep, a plurality of individual objects. However, we may (and often do) think of them—it—as a single object: a *herd* of sheep.[7] (Machover 1996, 10)

What has gone wrong in each of these cases is the attribution of inappropriate properties such as causal powers and location to abstract entities.[8] The problem is one of misappropriation. There are indeed concrete collections of sheep, grapes, apples, wolves, pigeons etc., into herds, bunches, piles, packs, flocks etc., but *none of them are sets*. They are concrete collectives with their own sorts of membership and persistence conditions, which vary widely and which need their own ontological treatment independently of mathematics. Typically they have causal powers and locations and non-extensional identity conditions. A somewhat different case are those collections of things which can be presented as sheer pluralities, for example by listing them, like Russell and Whitehead, or the several people who happen to be in a certain room now. Such pluralities, or manifolds as I once called them (Simons 1980), are what Russell called 'classes as many'

5. So one can be chased, attacked and even eaten by a set, oneself eat a set and absorb vitamins from it, press a set and make wine out of it, and have to clean up the droppings a set leaves on a statue.

6. So Juventus, Barcelona and Manchester United are very expensive sets.

7. So sets may safely graze on the field behind my house, and grow in the springtime by the addition of new members: lambs.

8. That it need not be so is magnificently demonstrated by Jech 1997, who unapologetically launches page 1 with axioms and then proceeds to deduce theorems: pure mathematics at its purest.

and Cantor called ‘Vielheiten’. The hegemony of set theory among philosophers concerned about collective entities has still to be broken. Among linguists, plurals have however gained a fair foothold, and the logician George Boolos persuasively championed pluralities as an ontologically economical way to interpret monadic second-order logic,[9] so the picture is not wholly bleak.

One effect of set theory in ontology has thus been to cripple the development of an adequate ontology of collective entities. This however is far from the worst of its effects. In general the employment of set theory, usually hand in hand with model-theoretic semantics, has been to persuade many philosophers that the rich panoply of entities the world throws at us can be reduced to individuals and sets of various sorts, for example sets as properties, sets of ordered tuples as relations, sets of possible worlds as propositions, and so on and so forth. It is hard to know where to start in revealing the scope of the damage caused to ontology by the thoughtless or supposedly scientifically economic reduction of various entities to sets. The most open-eyed proponent of this approach has been Quine, who to his credit has never shirked from confronting the issues, has never tried to sell sets under a false bill of fare, and is prepared to accept the absurdities as they arise, for example taking physical objects as sets of quadruples of real numbers, and thus as all composed of set-theoretic complications of the empty set. (Quine 1976) You, too, are a set, according to Quine, albeit a very complex one.

One area of ontology which has been beholden to set theory and has suffered from this is the philosophy of space and time, in particular the ontology of continua. Since the time of Bolzano, and especially since Cantor, it has been commonplace to regard continua as sets of points, in plain contradistinction to the mereological conception of geometrical entities assumed by Euclid. This has the following strange consequence, noticed and approved by Bolzano. If two bodies are continua, then they can only touch if one of them is topologically open and the other is topologically closed at the place or places of contact. Attempts by the more ontologically fastidious such as Brentano, Whitehead and Menger to account for continua otherwise than as treating them as sets of

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9. See the papers on plurals in Boolos 1998.

points have met with scant approval among mathematicians, and only marginally more among philosophers.

The wider damage done to ontology through the hegemony of predicate logic and its model-theoretic interpretations is discussed in Barry Smith's contribution to this volume and I shall not go into it.

Linguistics and philosophy of language

I shall however say something about the more linguistic end of logical semantics. The use of set theory in semantics goes back to Tarski and Carnap. Tarski's use of set theory to define satisfaction and truth for formal languages is rather sparing, and his ambivalent attitude to set theory comes over in his late work with Steven Givant which formulates it in an ontologically and ideologically parsimonious way. (Tarski and Givant 1986) The extensional attitude to logic which Tarski shared with other Poles, is of course wholly suited to the extensional framework of set theory. By contrast the other founder of logical semantics, Carnap, wished like his teacher Frege to give both extension and intension their due. He did so by what Ignacio Angelelli has with nice irony termed "the looking around method": Carnap writes in *Meaning and Necessity* that we must "look around" for entities to be the extensions and intensions of various classes of expressions. (Carnap ^{2}1956, 1) The traditional account of intension and extension was that whereas extensions (of predicates or terms) might be classes, intensions are properties. Set theory cannot represent intensions on its own. However a way to make intensions set-theoretically tractable was discovered, and it was Carnap who showed the way. It was to treat intensions as functions from possible worlds to extensions. Since functions are relations and relations are sets, this amounts to an extensionalization of intensional concepts. This approach to semantics was given its definitive form by Richard Montague, who added a host of other items such as times and places to the worlds to deal with indexical and other expressions. That set theory is the framework is left in no doubt: in his 1960 paper "On the Nature of Certain Philosophical Entities"[10] having "reduced" pains, events, tasks and obligations to predicates, and experiences to properties of two-place predicates (phenomenologists please note), Montague then adds (modestly):

> I have done no more than to reduce several dubious ontological categories to one, that of predicates; and one might well question the point of applying Occam's razor here. There are two reasons for the reductions. In the first place, predicates should not be regarded as wholly dubious. They are not much more mysterious than sets; and if we are willing to speak of possible worlds and possible individuals, we can say exactly what predicates are. […] The second reason […] is that it enables us now […] to construct an exact language capable of naturally accommodating discourse about the dubious entities. […] It has for fifteen years been possible for at least one philosopher (myself) to maintain that philosophy, at this stage in history, has as its proper theoretical framework set theory with individuals and the possible addition of empirical predicates. (Montague 1974, 152-154)

The following is a selection of quotes taken from a systematic book in the Montague style, which illustrate the extent to which set theory pervades the whole way of understanding language, and removes it from reality (Cresswell 1973; page numbers given with each passage):

> We want now to give a quite general definition of a *pure categorial language*. […] Where *Nat* is the set of all natural numbers 0, 1, 2 … etc. then the set *Syn* of *syntactic categories* is the smallest set satisfying:
>
> **5.11** Nat $\subseteq$ Syn
>
> **5.12** If $\tau, \sigma_1, \ldots, \sigma_n \in$ Syn then $\langle \tau, \sigma_1, \ldots, \sigma_n \rangle \in$ Syn (70f.)
>
> […]
>
> We can now define a pure categorial language $\mathscr{L}$ as follows. $\mathscr{L}$ is an ordered pair <F,E> where F is a function from Syn whose range is a set of pairwise disjoint finite sets of which all but finitely many are empty and the members of which are the *symbols* of $\mathscr{L}$ […] E is that function from Syn whose range is the system of smallest sets which satisfy the following conditions (71f.)
>
> […]
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> We shall say that a pure categorial language $\mathscr{L}$ is an *utterance language* iff its symbols are sets. By an *utterance* of a symbol α of $\mathscr{L}$ we mean simply a member of α. (87)

10. Republished in Montague 1974, 148–187.

In his introduction to Montague's papers, Richmond Thomason notes that Montague saw grammar as a branch of mathematics and not (as in Chomsky) of psychology. (Montague 1974, 2f.) Montague grammar as a piece of abstract theory is in fact rather interesting and even beautiful, but it is worth pulling ourselves up short and asking ourselves whether Montague has not in fact totally changed the subject; whether we need a framework consisting of an *n*-tuple of sets of possible worlds, moments in time and other contextual elements in order to make sense of the chatter of the man, woman or child in the street. Where is our sense of fitness and proportion if we do? It is a loss of that robust sense of reality enjoined on philosophers by Russell.

Sets in the philosophy of science

Set theory typically causes havoc when let loose in the philosophy of science. For example here is what Mark Steiner says about applying mathematics in physics:

> To "apply" set theory to physics, one need only add special functions from physical to mathematical objects (such as the real numbers). Functions themselves can be sets (ordered pairs, in fact). As a result, modern—Fregean—logic shows that the only relation between a physical and a mathematical object we need recognize is that of set membership. (Steiner 1999, 23)

In 1984 Philip Kitcher published an essay entitled "Species" in which he asserted, not unlike the set theorists quoted earlier, that biological species are sets of organisms, albeit ones having certain properties distinguishing them from arbitrary sets. Many of the absurd consequences of this view were immediately and patiently pointed out in the same issue of *Philosophy of Science* by Elliot Sober. (Kitcher 1984; Sober 1984)

The most radical disruption caused by letting set theory loose in the philosophy of science has been in that version of scientific structuralism associated with Joseph Sneed, and propagated through the German speaking world by Wolfgang Stegmüller. Two modern exponents of Sneed's views, Balzer and Moulines, list the specific notions of the Sneedian programme as follows (Balzer and Moulines 1996, 12f., quoted from Schmidt 2003):

M_p	A class of potential models (the theory's conceptual framework)
M	A class of actual models (the theory's empirical laws)
$<M_p, M>$	A model-element (the absolutely necessary portion of a theory)
M_{pp}	A class of partial potential models (the theory's relative non-theoretical basis)
C	A class of constraints (conditions connecting different models of one and the same theory)
L	A class of links (conditions connecting models of different theories)
A	A class of admissible blurs (degrees of approximation admitted between different models)
$K = <M_p, M, M_{pp}, C, L, A>$	A core (the formal-theoretical part of a theory)
I	The domain of intended applications ("pieces of the world" to be explained, predicted or technologically manipulated)
$T = <K,I>$	A theory-element (the smallest unit to be regarded as a theory)
σ	The specialization relation between theory-elements
N	A theory-net (a set of theory-elements ordered by σ—the "typical" notion of a theory)
E	A theory-evolution (a theory-net "moving" through historical time)
H	A theory-holon (a complex of theory-nets tied by "essential" links)

It is not that these factors are not relevant for a rounded and realistic treatment of scientific theories and their applications. Rather what is strange is the assumption that a science is not properly understood or grounded unless it has been, as Stephan Körner was wont to say, *sneedifiziert*, that is, cast in this model-theoretic and hence set-theoretic form. In all of the above components the only mention of the world, reality, things or real phenomena is in the domain, and it is extremely suspicious that the term 'pieces of the world' occurs in scare quotes. Sneedism consists in the total or almost total replacement of the real world by a set-theoretic concoction, and is thus the fitting *wissenschafts-theoretisches* counterpart to Montaguism in linguistics and model theory in ontology. How anyone can call this a realistic or sensible view of science is totally beyond me.

Abstinence from sets

It would be tiresome to continue citing further absurdities in philosophy resulting from the over-zealous application of set theory. The outstanding question which a disillusioned but anxious analytic philosopher may legitimately ask is how he or she could manage without sets. Can one survive without them? My advice is to be of good cheer: you will manage perfectly well. Start by a regime of replacing expressions roughly as follows:

Replace	**By**
[Names of sets]	[Plurals, collective terms]
set	class, collection, plurality
the set of A	the A, all A
a set of A	some A, several A
a subset of A	some (of) A
element of A	one of A
A is a subset of B	all A are B
union of A and B	things which are A or B
intersection of A and B	things which are both A and B
power set of A	all collections of As
universal set/class	everything
empty set	nothing [or avoid altogether]
the intersection of A and B is empty	no A is B
{x}	x

{x,y}	x and y
the ordered pair <x,y>	x followed by y; x and y in that order
set of ordered pairs	relation
a theory is an *n*-tuple <…,…>	a theory has several features, as follows …

For those who are heavily addicted to decorating their papers and presentations with set terminology, the process of withdrawal ought to be gradual and progressive, starting with, say, set-free Fridays or a period of abstinence during Lent. They must be prepared for a certain loss of surface dazzle because the replacement of the impressive and technical sounding set-theoretic vocabulary by plain plurals and ordinary words will lose them some of the hard gloss that can make philosophy sound more like real science. The reward for this abstinence will be greater honesty, transparency and accessibility.

Simply lacking quick and easy ways to put certain things may of itself force one to re-examine old ideas and reveal some of them as prejudices. For example, it was long an article of faith among philosophers of science that a theory is a set of sentences or propositions closed under logical consequence, i.e. if there are propositions *P* in the theory and they logically entail a proposition *q*, then *q* is in the theory. As standardly understood, this means that every theory consists of infinitely many propositions. This stands in crass contrast to the reality, which is that all theories are actually finite but are capable of extension, for example by adding logical consequences of propositions already in the theory as these are discovered by deduction. Simply by refusing glibly to identify theories with sets of propositions one takes the first fruitful step on the road to a more realistic and more adequate understanding of what theories are and how they work. (Obviously this does not mean we should give our money to the Sneedy.)

Another not unimportant example is this. The basis of nearly all modern theories of truth is Tarski. Tarski's theory schema was expressly formulated for, and only for, what he called 'the deductive sciences', i.e. logic and mathematics. Though less ontologically exuberant than Carnap's semantics, Tarski's still adopts a Platonistic attitude to truth-bearers (which he takes, not without some misgivings, as sentence-types), because he needs to explain how proofs work in mathematics (and he says as much). There have to be infinitely many such sentence-types. If we take a more realistic and indeed naturalistic view of the bearers of truth and falsity, namely seeing them as individual token judgings, believings, assertings, and so on, then we are forced to confront the question how truth and falsity work *without* the simplifying assumptions that Tarski needed to make in order to deliver the strong metamathematical results he wanted. A philosopher ought to want to explain how truth and falsity work in real life, not in the exceptional case of mathematics.

The proper method of ontology is not that of set-theoretic construction or reduction. The ontologist, like the geographer, can only discover what is there and give it a name. We start by hunting and gathering specimen entities for ontology, from any source whatever. Then the theoretical weighing and sifting can begin, finding out whether we can or cannot do without certain putative things, such as universals, continuants, events, states of affairs, mathematical objects, immaterial souls, God, and so on. The ultimate aim is a universal and systematic taxonomy in which every item is an instance of a category. There *may* be a place in that ultimate taxonomy for sets. But it will not be pride of place.

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