

Vague Reference and Approximating Judgments

Thomas Bittner

Department of Computer Science, Northwestern University, bittner@cs.nwu.edu

Barry Smith

Department of Philosophy, State University of New York at Buffalo, phismith@buffalo.edu

Abstract: We propose a new account of vagueness and approximation in terms of the theory of granular partitions. We distinguish different kinds of crisp and non-crisp granular partitions and we describe the relations between them, concentrating especially on spatial examples. We describe the practice whereby subjects use regular grid-like reference partitions as a means for tempering the vagueness of their judgments, and we demonstrate how the theory of reference partitions can yield a natural account of this practice, which is referred to in the literature as ‘approximation’.

1. INTRODUCTION

Consider the proper name ‘Mount Everest’. This refers to a mereological whole, a certain giant formation of rock. A mereological whole is the sum of its parts, and Mount Everest certainly contains its *summit* as part. But it is not so clear which parts along the foothills of Mount Everest are parts of the mountain and which belong to its neighbors. Thus it is not clear which mereological sum of parts of reality actually constitutes Mount Everest. One option is to hold that there are multiple candidates, no one of which can claim exclusive rights to serve as the referent of this name. Each of these many candidates has the summit, with its height of 29,028 feet, as part. These candidates are all perfectly determinate portions of reality. They differ, however, regarding which parts along the foothills are included as parts and which are not.

Varzi (2001) refers to the above as a *de dicto* view of vagueness. It treats vagueness not as a property of objects but rather as a semantic property of names and predicates, a property captured formally in terms of a supervaluationistic semantics (Fraassen 1966), (Fine 1975). We shall concentrate our attentions in what follows on the case of *singular reference*, i.e., reference via names and definite descriptions to concrete portions of reality such as mountains and deserts. We shall also concentrate primarily on spatial examples. As will become clear, however, it is one advantage of the framework here defended, that it can be generalized automatically beyond the spatial case.

In order to understand vague reference we use the theory of granular partitions we advanced in our earlier papers (Smith and Brogaard to appear), (Bittner and Smith 2001), (Smith and Bittner 2001). The fundamental idea is that every use of language to make a judgment about reality brings about a certain *granular partition*. We define granular partitions as systems of cells conceived as projecting onto reality in something like the way in which a bank of flashlights projects onto reality when it carves out cones of light in the darkness. A judgment, J , is then a pair, consisting of a sentence, S , and an associated granular partition GP , i.e., $J = (S, GP)$.

We consider reference as a *two-step-process*. Language tokens are associated with cells in a grid-like structure, and these cells are projected onto reality in the way suggested by our flashlight metaphor. (Bittner and Smith 2001) and (Bittner and Smith 2001) show how this two-step-process allows us to explain the features of selectivity and granularity of reference in judgments. In this paper we show how the same machinery can help us to understand the phenomena of vagueness and approximation.

2. CRISP AND VAGUE GRANULAR PARTITIONS

The theory of granular partitions has two parts: (A) a theory of the relations between cells and the structures they form, and (B) a theory of the relations between cells and objects in reality. Consider Figure 1. The left part shows a cell structure, with cells labeled *Everest*, *Lhotse* and *The Himalayas*. The right part shows portions of reality onto which those cells project.



Figure 1: Left: a partition, with cells *Lhotse*, *Everest* and *The Himalayas*. Right: A part of the Himalayas seen from space, with admissible candidate referents for ‘Mount Lhotse’ (left) and ‘Mount Everest’ (right).

We define the cell structure, A , of a partition, P_t , as a system of cells, z_0, z_1, \dots . We write $Z(z, A_{P_t})$ as an abbreviation for ‘ z is a cell in the cell-structure A of the partition P_t ’. We say that z_1 is a *subcell* of z_2 if the two cells are in the same cell structure and the first is contained in the latter, and we write $z_1 \subseteq_A z_2$ in order to designate this relationship. (In the remainder we omit subscripts wherever the context is clear.) We then impose four axioms (or ‘master conditions’) on all partitions, as follows:

- MA1: The subcell relation \subseteq is reflexive, transitive, and antisymmetric.
- MA2: The cell structure of a partition is always such that chains of nested cells are of finite length.
- MA3: If two cells overlap, then one is a subcell of the other.
- MA4: Each partition contains a unique maximal cell.

These conditions together ensure that each partition can be represented as a tree (a directed graph with a root and no cycles).

Theory (B) arises in reflection of the fact that partitions are more than just systems of cells. They are constructed in such a way as to project upon reality as indicated in Figure 1. When projection is successful, then we say that the object targeted by a cell is located in that cell. We then write ‘ $P(z, o)$ ’ as an abbreviation for: cell z is projected onto object o , and ‘ $L(o, z)$ ’ as an abbreviation for: object o is located in cell z . We call partitions *transparent* if and only if MB1 and MB2 hold:

- MB1 $L(o, z) \rightarrow P(z, o)$.
- MB2 $P(z, o) \rightarrow L(o, z)$.

We demand further that projection and location be functional relations, i.e., that every cell projects onto just one object and every object is located in just one cell:

- MB3 $P(z, o_1) \text{ and } P(z, o_2) \rightarrow o_1 = o_2$
- MB4 $L(o, z_1) \text{ and } L(o, z_2) \rightarrow z_1 = z_2$

The partitions of interest in this paper are *complete* in the sense that every cell projects onto at least one object, i.e., there are no empty cells (no cells projecting outwards into the void):

- MB5 $Z(z, A) \rightarrow \exists o: L(o, z)$

The resulting class of partitions is quite narrow. For a more general treatment, embracing also not so well-behaved granular partitions, see (Bittner and Smith 2001).

A *vague* granular partition $Pt^v = (A, P^v, L)$ is a triple such that: A is a system of cells for which MA1-4 hold, L is a location relation as described above, and P^v is a *class* of projection relations. Since reality is not vague (in our *de dicto* view) objects are located in cells in the same way as in the crisp case. However there are in the vague case *classes of projection* relations, each targeting different candidates of vague reference.

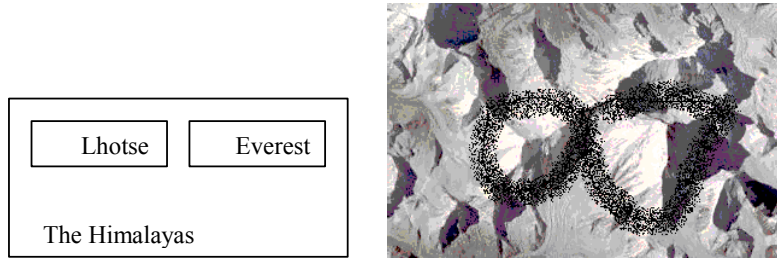


Figure 2: Left: a partition, with cells *Lhotse*, *Everest* and *The Himalayas*. Right: A part of the Himalayas seen from space, with cloudy ovoids representing the family of admissible candidate referents for ‘Mount Lhotse’ (left) and ‘Mount Everest’ (right).

Consider Figure 2, which depicts a vague partition $Pt^v = (A, P^v, L)$ of the Himalayas. This has a cell structure according to theory A , as shown in the left part of the figure. On the right is depicted a multiplicity of possible candidate projections for the cells in A , indicated by boundary regions depicted via cloudy ovoids. The boundaries of the actual candidates onto which the cells ‘Lhotse’ and ‘Everest’ are projected under the various P_i in P^v are included somewhere within the clouds of regions depicted in the figure.

We now introduce a relation $AP(o', o)$, with the reading: o' *approximates* o . Notice that the variables o, o' refer in every case to crisp entities (reflecting our *de dicto* view of vagueness, according to which vagueness arises only at the level of terms and is not a feature of the reality to which our terms refer). In the case of crisp reference AP is just the identity relation; here the only entity approximating o is o itself. In the case of vague reference AP gives rise to an equivalence relation that holds among a plurality of admissible candidate referents, as follows:

$$o' \approx_{AP} o'' \equiv \exists o: AP(o, o') \text{ and } AP(o, o'').$$

This plurality reflects the fact that, in the case of a term like ‘Mount Everest’, there is no fact of the matter that singles out which specific candidate referent is the object in question: all admissible candidate referents are equally good approximations. More specifically, there are no necessary and sufficient conditions that would allow a judging subject to identify exactly the portions of reality that constitute Everest. All the candidate referents indicated by the cloudy ovoids in Figure 2 are equivalent in the sense of \approx_{AP} .

We demand that if the object o is located in the cell z , then there is a projection P_i that targets a portion of reality o' which is a candidate referent of o .

$$\text{MB1}^V \quad L(o, z) \rightarrow \exists i \exists o': P_i(z, o') \text{ and } \text{AP}(o', o)$$

We also demand that, if there is a projection P_i which targets a portion of reality o' which is a candidate referent of the object o , then o is actually located in the cell z .

$$\text{MB2}^V \quad \forall i: P_i(z, o') \text{ and } \text{AP}(o', o) \rightarrow L(o, z)$$

In the context of this paper MB1^V and MB2^V can be simplified as:

$$\forall i: (P_i(z, o') \text{ and } \text{AP}(o', o)) \leftrightarrow L(o, z).$$

We also demand that L and all the P_i are functional in the sense discussed in the crisp case:

$$\text{MB3}^V \quad P_i(z, o_1) \text{ and } P_i(z, o_2) \rightarrow o_1 = o_2$$

$$\text{MB4}^V \quad L(o, z_1) \text{ and } L(o, z_2) \rightarrow z_1 = z_2$$

We demand further that cells project onto some object under every projection:

$$\text{MB5}^V \quad Z(z, A) \rightarrow \exists o: L(o, z)$$

and that the targets of all projections for any given cell are equivalent in the sense of \approx_R .

$$\text{MB6}^V \quad P_i(z, o_1) \text{ and } P_j(z, o_2) \rightarrow o_1 \approx_{\text{AP}} o_2.$$

These conditions merely summarize in formal terms what is involved in the thesis that, when we refer to 'Mount Everest', our reference projects vaguely across a certain area of reality.

3. BOUNDARIES THAT DELIMIT VAGUENESS

Consider Figure 1. The cells labeled 'Everest' and 'Lhotse', carve *mountain-candidates* out of a certain formation of rock. They do not do this physically, but rather by establishing fiat boundaries in reality, represented by the black lines in the right part of the figure. (Smith 1995), (Smith 2001), (Bittner and Smith 2001). But how are judging subjects able to impose or specify spatial boundaries *vaguely*, along the lines indicated in Figure 2?

Suppose you are hiking through the Himalayas with your friends and you make the judgment:

[A]: We will cross the boundary of Mount Everest within the next hour.

Through your use of the phrase 'within the next hour' you delimit a *range* of admissible candidates for the boundary of Mount Everest along the trajectory of your hike. Consider the left part of Figure 3. Boundaries delimiting admissible candidates are imposed by specifying a time interval that translates to travel distance along a path; time serves here as frame of reference. The boundaries are defined by the current location of the judging subject (marked: 'now') and his location after the

specified time has passed (marked: ‘in one hour’). The boundary of each admissible candidate referent crosses the path at some point between these two boundaries. In general we call the first boundary the *exterior boundary* and the second the *interior boundary*.

The general case is illustrated in the right part of Figure 3, which is intended to depict how, in the process of approximation, judging subjects project egg-yolk-like granular partitions onto reality involving three cells: an exterior, a core, and an intermediate region within which the boundary candidates are. See (Cohn and Gotts 1996) and (Roy and Stell to appear) This granular partition serves as the frame of reference in terms of which the judging subject is able to both specify and constrain the range of admissible entities to which he (vaguely) refers.

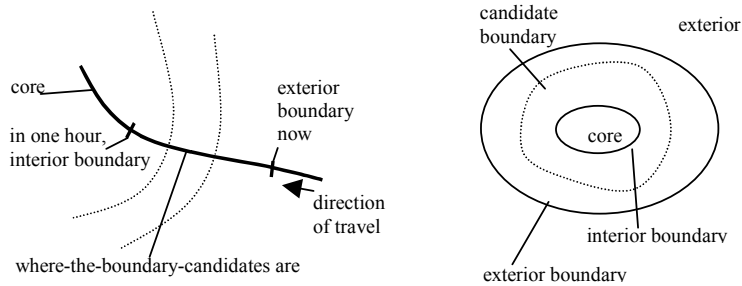


Figure 3: Boundaries that limit vagueness

Besides the case where new fiat boundaries are imposed in order to delimit vagueness, as discussed above, there are cases where already existing bona fide or fiat boundaries are re-used for the same purpose. For example, there is one crisp granular partition with which we are all familiar. It has exactly 50 cells, which project onto the 50 constituent states of the United States of America. A fragment of this partition is presented in the left and right parts of Figure 4. In the foreground of the figure we see in addition an area of bad weather, represented by a dark dotted region that is subject to vagueness *de dicto* in the sense discussed above. Wherever the boundaries of this object might be located, they certainly lie skew to the boundaries of the relevant states. But the figure also indicates (with the help of suitable labeling) that there are parts of the area of bad weather that are also parts of Wyoming, others which are parts of Montana, others which are parts of Utah, and yet others which are parts of Idaho.

In the sorts of contexts – represented by more or less coarse-grained partitions – which we humans normally inhabit, it is impossible to refer to any *crisp* boundary when making judgments about the location of a bad weather region of the sort described. However it is possible to describe its (current) location relative to something like the grid of a map including, in our case, the underlying US-state partition:

[B] The area of bad weather extends over parts of Wyoming, parts of Montana, parts of Utah, and parts of Idaho.

We, the judging subjects, then deliberately employ this partition as our frame of reference and we describe the *relationships* that hold between all admissible referents of the vague term ‘area of bad weather’ and the cells of this partition. In terms of spatial relations this means in the given case that all admissible candidates partially overlap the states of Wyoming, Montana, Utah, and Idaho and that they do not overlap any other state. Consequently, if a judging subject can specify for every partition cell a unique relation – for example *part of* – that holds for all admissible candidate referents of a vague term, then this is a *determinate way to effect vague reference*. A meteorologist may achieve a finer approximation by employing a finer-grained partition as frame of reference in order to make a more specific judgment about the current location of the bad weather region. Thus she might use cells labeled Eastern Idaho, Southern Montana, Western Wyoming, and Northern Utah, and so on. The latter yield a fiat boundary of the sort depicted in the right part of Figure 4.

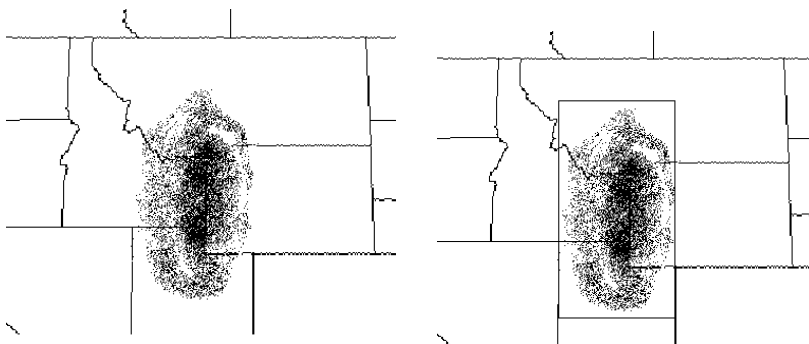


Figure 4: States of the United States with a bad weather system

Notice that all these boundaries existed already before the judgments which use them as frame of reference were made. They are only re-used in order to formulate constraints on the possible location of admissible candidates of the corresponding vague referring terms. Judging subjects re-use existing boundaries in this way in order to make determinate judgments about approximate locations. They do so because this is a convenient and determinate way to make vague reference, and it has even greater utility when the frame of reference is a commonly accepted one, as in the present case.

4. APPROXIMATING JUDGMENTS

Approximating judgments are expressed by sentences like: [A] ‘We will cross the boundary of Mount Everest within the next hour’, [B] ‘The area of bad weather extends over parts of Wyoming, parts of Montana, parts of Utah, and parts of Idaho’.

Approximating judgments are a special class of judgments that contain both vague names and, in addition, a (relatively) crisp reference to boundaries that

delimit this vagueness. Consider the approximating judgment [A]. This judgment is vague because it contains the vague name ‘Everest’. It is an approximating judgment since it contains the reference to boundaries delimiting the vagueness of the name ‘Everest’ by referring to boundaries that delimit admissible candidate referents for ‘Everest’ via the phrase ‘within the next hour’. In this paper we consider approximating judgments which contain a single vague name and a crisp reference frame. More complex cases are possible – including cases where the reference frame itself involves a certain degree of vagueness – but formal consideration of these is omitted here since their treatment follows the same basic pattern.

An approximating judgment J^A , if uttered successfully, imposes *two* partitions onto reality: a vague partition Pt^V , representing the supervaluationist semantics for the vague name N , and a *reference partition* Pt^R , which delimits the vagueness with which N refers. We say that an *approximating judgment* J^A is a triple (S, Pt^V, Pt^R) , consisting of a sentence, S , together with two granular partitions, Pt^V and Pt^R .

Consider, for example, the approximating judgment $J^A_F = ([A], Pt^V, Pt^R)$, expressed by the sentence [A] containing the vague name ‘Everest’ and the constraint on admissible boundaries of admissible candidates of ‘Everest’ that is expressed by the phrase ‘within the next hour’. The vague name ‘Everest’ imposes a vague partition Pt^V with a corresponding cell labeled ‘Everest’ projecting onto the multiplicity of admissible candidate referents of the name ‘Everest’, as indicated in Figure 2.

The judgment [A] imposes a *reference partition* Pt^R onto reality, as depicted in the left part of Figure 3. Intuitively, this reference partition projects onto reality in a way that constrains admissible candidate referents for ‘Everest’, i.e., it constrains admissible projections of the cell ‘Everest’ in the vague partition Pt^V . The latter must all be such that they will be crossed by the judging subject between now and one hour hence.

Another example of an approximating judgment is $J^A_K = ([B], Pt^V, Pt^R)$, expressed by the sentence: [B] ‘The area of bad weather extends over parts of Wyoming, parts of Montana, parts of Utah, and parts of Idaho’. The corresponding vague partition Pt^V contains a cell labeled ‘the area of bad weather’ projecting onto a multiplicity of admissible candidates. The judgment J^A_K reuses the partition depicted in the right part of Figure 4 as reference partition Pt^R . The latter constrains admissible projections of the cell labeled ‘the area of bad weather’ in Pt^V in such a way that each candidate referent that is targeted by a projection P_i of Pt^V must extend over parts of reality targeted by the cells of Pt^R labeled ‘Wyoming’, ‘Utah’, ‘Montana’, and ‘Idaho’. Moreover Pt^R implicitly constrains admissible projections of the cell ‘the area of bad weather’ in such a way that no candidate referent targeted by a projection P_i of Pt^V can extend over parts of reality – for example Hawaii – targeted by cells of Pt^R with labels not mentioned in [B].

We now continue by giving a formal partition-theoretic definition of reference partitions of the sort described.

5. PARTITION THEORY AND APPROXIMATION

The idea underlying the partition-theoretic view of approximation is that a (crisp) granular partition can be used as a *frame of reference* (a generalized

coordinate frame (Bittner 1997)), which allows us (a) to describe the *approximate location* of objects and (b) to project onto portions of reality in an approximate way. We call a granular partitions which is used as a frame of reference a *reference partition*.

Consider a vague name such as ‘The hurricane Walter’ and the corresponding multiplicity of admissible candidates formed by crisp portions of reality in the domain of the Northwestern United States. Consider another crisp partition structuring this same domain but without recognizing any of the candidates referred to by the name ‘Walter’ directly. This might be a partition working with the boundaries of federal states, as in Figure 4, or a partition formed by a raster of cells aligned to lines of latitude and longitude (as in the right part of Figure 5). Such a reference partition can be used in order to approximate the multiplicity of admissible candidate referents in a uniform and determinate manner.

To see how this works, we introduce the three concepts of full overlap (*fo*), partial overlap (*po*), and non-overlap (*no*), concepts which we shall use to generalize the notions of projection and location, as follows. Let *o* be a portion of reality that is not directly located in a cell of a given reference partition and let *x* be a portion of reality that is located at the cell *z* of our reference partition (the cell *z* projects onto *x*). *x* is, in the cases mentioned, a region of space on the surface of the Earth. The constants *fo*, *po*, *no* will now be used to measure the degree of mereological coverage of the object *x* by the object *o*.

We call the relation $L^R(o, z, \omega)$ the *rough location* of the portion of reality *o* with respect to the cell *z* and the relation $P^R(z, o, \omega)$ the *rough projection* of the cell *z* onto *o*. (We use the phrases ‘rough location’ and ‘rough projection’ in order to emphasize the correspondence to the notion of rough approximation using rough sets advanced in (Pawlak 1982).) In both relations the value ω characterizes the degree of mereological overlap of the portion of reality targeted by the cell *z* with the actual portion of reality *o*, i.e., it takes the value *fo*, *po*, or *no*. Consider the left part of Figure 4. There the relation *po* holds between all admissible candidate referents (BWA_i) of ‘the area of bad weather’ and Montana, i.e., $\forall i: L^R(BWA_i, Montana, po)$. The relation *no* holds between all the BWA_i and Oregon, i.e., $\forall i: L^R(BWA_i, Oregon, no)$.

We can characterize the relationships between exact and rough location and exact and rough projection in reference partitions as follows:

$$\begin{aligned} L^R(o, z, fo) &\equiv \forall x: L(x, z) \rightarrow x \leq o \\ P^R(z, o, fo) &\equiv \forall x: P(z, x) \rightarrow x \leq o \\ L^R(o, z, po) &\equiv \forall x: L(x, z) \rightarrow \exists y: y \leq x \text{ and } y < o \\ P^R(z, o, po) &\equiv \forall x: P(z, x) \rightarrow \exists y: y \leq x \text{ and } y < o \\ L^R(o, z, no) &\equiv \forall x: L(x, z) \rightarrow \neg \exists y: y \leq x \text{ and } y < o \\ P^R(z, o, no) &\equiv \forall x: P(z, x) \rightarrow \neg \exists y: y \leq x \text{ and } y < o \end{aligned}$$

The notion of rough location gives rise to an equivalence relation in the domain of objects (portions of reality), with respect to a given reference partition Pt with rough location relation L^R and rough projection P^R , as follows:

$$o_1 \approx_{Pt} o_2 \equiv \forall z: L^R(o_1, z, \omega) \leftrightarrow L^R(o_2, z, \omega).$$

Two objects, in other words, are equivalent with respect to the granular partition Pt^R if and only if they have an identical rough location with respect to all cells of

this partition. \approx_{Pt} can thus be interpreted as meaning indiscernibility with respect to the frame of reference provided by the partition Pt^R . In an approximating judgment $J = (S, Pt^V, Pt^R)$ the reference partition Pt^R will be chosen in such a way that the candidate referents targeted by a single cell in Pt^V are equivalent with respect to \approx_{Pt} .

We define a *reference partition* as a fourtuple, $Pt^R = (A, P^R, L^R, \Omega)$ where A is a cell structure, P^R and L^R are a rough projection and location relations, and Ω is the set of values (*fo*, *po*, *no*), indicating the degrees of overlap distinguished (coarser and finer distinctions are possible, as discussed in (Bittner and Stell 1998)). The equivalence relation \approx_{Pt} is given indirectly by L^R and Ω . We then demand that the following counterparts of MB1–4 hold for reference partitions:

$$\begin{aligned} MB1^R & L^R(o, z, \omega) \rightarrow P^R(z, o, \omega) \\ MB2^R & P^R(z, o, \omega) \rightarrow L^R(o, z, \omega) \\ MB3^R & \forall o: (L^R(o, z_1, \omega) \text{ and } L^R(o, z_2, \omega)) \rightarrow z_1 = z_2 \\ MB4^R & \forall z: (P^R(z, o_1, \omega) \text{ and } P^R(z, o_2, \omega)) \rightarrow o_1 \approx_{Pt} o_2 \end{aligned}$$

MB3^R tells us that if two cells are such that all objects cover the targets of these cells in the same way, then these cells must be identical. MB4^R tells us that if two objects have the same relations (*fo*, *po*, *no*) to all cells of a granular partition Pt then the two objects are indiscernible with respect to this partition.

If an approximating judgment like $([A], Pt^V, Pt^R)$ is to succeed, then the reference partition needs to project onto reality in such a way that all admissible candidate referents are equivalent with respect to the indiscernibility relation imposed by Pt^R . Thus, in the hiker case, each value of p_i^V ('Everest') must be such that its border can be crossed in one hour from the time when the judgment is made.

Let (S, Pt^V, Pt^R) be an approximate judgment. Let Pt^V be a vague partition with a cell for each vague name in S . We then demand that the reference partition Pt^R must be such that for every cell in Pt^V the object located in this cell and all its candidate referents are equivalent with respect to Pt^R :

$$MB5^R \quad (Z(z, Pt^V) \text{ and } L(x, z)) \rightarrow \forall i: p_i(z) \approx_{Pt} x$$

From this it follows that candidate referents equivalent in the sense of \approx_{AP} are also equivalent in the sense of \approx_{Pt} , or in other words:

$$o1 \approx_{AP} o2 \rightarrow o1 \approx_{Pt} o2.$$

Notice that the reverse direction of the implication does not hold, since there might be objects that are equivalent in respect to the reference partition but which are not candidate referents approximating the same object.

6. CONSTRAINING APPROXIMATION

For each cell z in a vague partition Pt^V we can now classify corresponding portions of reality into three classes: core parts with respect to z , boundary parts with respect to z , and exterior parts with respect to z . We say that:

1. x is part of the *core* of the object which is located in the cell z under the vague projection P^V if and only if, under *all* projections $p_i(z)$ in P^V , x is a part of the targeted candidate referent. In symbols:

$$\text{core}_V(x, z) \equiv \forall i: x \leq p_i(z)$$

2. x is a part of the *boundary* of the object which is located in the cell z under projection P^V if and only if there are some projections p_i in P^V under which x is part of the targeted candidate referent and other projections p_j in P^V under which this is not the case:

$$\text{boundary}_V(x, z) \equiv \exists i: x \leq p_i(z) \text{ and } \exists j: \neg(x \leq p_j(z))$$

3. x is part of the *exterior* of the reference to the object onto which is located in cell z under projection P^V if and only if x is not reached by any projection p_i in P^V of this cell.

$$\text{exterior}_V(x, z) \equiv \forall i: \neg(x \leq p_i(z))$$

(For example, there are parts of reality – such as Berlin – that are not reached by any of the projections of the cell ‘Everest’ in any natural context.)

Reference via a vague name creates a partition of reality into core, boundary, and exterior parts, where $\text{CORE}_V(z)$, $\text{BOUND}_V(z)$, and $\text{EXT}_V(z)$ are the mereological sums of all core, boundary, and exterior parts of reality with respect to the vague projection of the cell z . We define the *semantic partitioning* of reality with respect to a vague name N to be a partitioning of this sort generated in reflection of the vagueness of N .

We can now distinguish between approximations that exactly target all candidates of reference (and nothing else) and constraining approximations. Consider the approximating judgment $J^A_{F=}([A], \text{Pt}^V, \text{Pt}^R)$. The reference partition Pt^R (shown in the left part of Figure 3) imposes two fiat boundaries onto reality: an interior boundary of the approximation and an exterior boundary of the approximation. As discussed above, this often results in a partition structure similar to the one depicted in the right part of the figure. The (geometric) projection of this partition onto the path the judging subject takes on her journey towards the summit of Mount Everest results in the reference partition Pt^R .

Consider now the semantic partition imposed by the cell labelled ‘Everest’ in the vague partition Pt^V . One can see that the relationship between Pt^V and Pt^R satisfies MB5^R , which means that all candidate referents are equivalent under \approx_{Pt} . Consider now the location of two pairs of boundaries: (a) the interior and exterior boundaries imposed by the judging subject as a frame of reference for her approximation; and (b) the boundaries imposed by the semantic partitioning into core, boundary, and exterior via the cells of Pt^V . We say that the approximating judgment exactly targets all candidates of reference for the name N if and only if (1) the interior boundary of the approximation coincides with the boundary separating the core from the surrounding parts of the semantic partition; and (2) the exterior boundary of the approximation coincides with the boundary separating exterior parts from what lies within. In this case two objects have the same approximation if and only if they are candidate referents approximating the same object:

$$AP(o_1, o) \text{ and } o \approx_{Pt} o_1 \rightarrow \forall o_2: o_1 \approx_{Pt} o_2 \leftrightarrow o_1 \approx_{AP} o_2.$$

An approximating judgment that does not have these properties is called *imprecise*. Consider Figure 4. One can see that there are parts of reality, say x , which in addition to the parts covered by candidate referents of ‘The hurricane Walter’ also covers the very Northwestern parts of Idaho. In this case we have $x \approx_{Pt} o_2$ but $\neg(x \approx_{AP} o_2)$. For this reason the approximation of the vague reference of the name ‘The hurricane Walter’ is imprecise. Examples for approximation that exactly target all candidates of reference (and nothing else) can be found in fiat domains where objects are brought into existence by demarcation using a wide boundary as indicated in the right part of Figure 3 rather than by demarcation using a single crisp boundary in the sense of (Smith 1995).

Consider now more complex reference partitions, like the political subdivision of the United States used in judgment [B] (Figure 4). In order to compare the semantic partitioning imposed by the vague reference to the area of bad weather we need to consider mereological sums of partition cells rather than partition cells taken singly. The interior boundary of the approximation coincides with the boundary of the mereological sum of targets of partition cells for which the relation fo holds. In our example this sum is empty. (In the left part of Figure 5 the interior boundary of the approximation coincides with the boundary of the cell K.) The exterior boundary of the approximation coincides with the boundary of the mereological sum of targets of partition cells for which the relation po holds. In the left part of Figure 4 this boundary is the boundary of the mereological sum of the states Wyoming, Montana, Idaho, and Utah. (In the left part of Figure 5 the outer boundary of the approximation coincides with the outer boundary of the mereological sum of the cells [A,..., P] minus the cells A and M.) The interior and exterior boundary of an approximation therefore corresponds to the boundaries of the mereological sums of the lower (interior boundary) and upper (exterior boundary) approximations in the sense of rough set theory:

$$\begin{aligned} \text{Lower}(o) &= +_{p(z)} P^R(z, o, fo) \\ \text{Upper}(o) &= +_{p(z)} (P^R(z, o, fo) \text{ or } P^R(z, o, po)) \end{aligned}$$

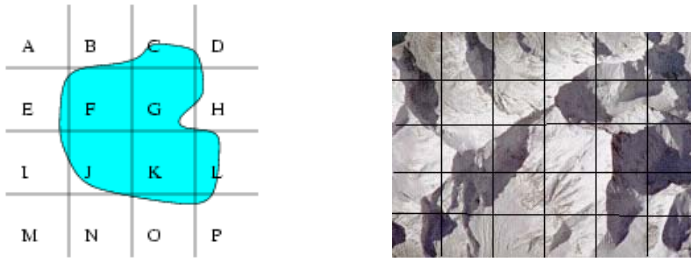


Figure 5: Rough approximation

Due to the nature of vagueness most approximations will be imprecise, but some belong to the class of what we shall call *constraining* approximations. Let o be an object in reality subject to vague reference, let z be the cell in the vague partition Pt^V

in which o is located. The approximation of o with respect to the approximating partition Pt^R is called constraining if and only if the following holds:

$$\text{Lower}(o) \leq \text{CORE}_v(z) \leq (\text{CORE}_v(z) + \text{BOUND}_v(z)) \leq \text{Upper}(o)$$

Consider the class of constraining approximations. As discussed in (Bittner and Smith 2001), judging subjects will use approximating judgments which are constraining and are *as precise as necessary* in whatever is the context in hand. This means that the limits imposed on the vagueness by the approximation will be such that the resulting judgment is not subject to truth-value indeterminacy. The judgments we actually make in normal contexts (as contrasted with those types of artificial judgments invented by philosophers) are determinately either true or false even in spite of the vague terms which they contain.

7. PROPERTIES OF REFERENCE PARTITIONS

Every reference partition Pt^R has a *skeleton* $Pt^S = (A, P^S, L^S)$, which is a crisp granular partition with the following properties:

- i. P^S is the restriction of P^R to triples of the form (z, o, fo) for objects o that are recognized directly by single cells z in A ;
- ii. A is identical to the cell structure of Pt^R ;
- iii. L^S is the restriction of L^R to triples of the form (o, z, fo) for objects o that are recognized directly by single cells z in A ;
- iv. Pt^S satisfies MA1–4, MB1–5.

It is the skeleton of the reference partition that actually establishes the frame of reference for the approximation, i.e., projects onto the boundaries that constrain or delimit the vagueness involved.

Consider the judgment ‘The area of bad weather extends over parts of Wyoming, parts of Montana, parts of Utah, and parts of Idaho’ and the corresponding formal structure $J^A = ([B], Pt^V, Pt^R)$. The skeleton of the reference partition Pt^R is the partition Pt^S , which recognizes the federal states of the US (Figure 4) and thereby establishes the frame of reference for the approximation. Consider Figure 3. The skeleton s of the reference partitions contains the cells ‘core’, ‘exterior’, and intermediary zone (where the boundary candidates are); this cell structure is shared in both cases with the reference partition Pt^R . The *crisp* projection (Pt^S) of these cells establishes the (interior and exterior) boundaries which limit the vagueness along the lines discussed above.

The skeletons of reference partitions which serve as frames of reference are often spatial or temporal in nature. They characteristically have the following properties:

- (1) they are relatively stable, i.e., they do not change over time (we can also demand that they are specifiable in some easily communicable way);
- (2) they preserve mereological structure;
- (3) they do not contain empty cells (every cell projects onto some object);
- (4) the mereological sum of the relevant minimal cells is identical to the root cell (hence reference partitions do not contain ‘empty space’ (in

- contrast, say, to the periodic table, which contains cells kept in reserve to project onto elements yet to be discovered);
- (5) the mereological sum of the targets of all partition cells is identical to the domain of the partition; in symbols:

$$CE \quad \exists z, z_1, \dots, z_n: \text{Max}(z) \text{ and } p(z) = +_{\text{Min}(z_i)} p(z_i),$$

where $p(z)$ returns the object onto which z is projected.

(1) implies in turn: (a) that the pertinent cell structure is fixed and (b) that the objects onto which the skeleton projects do not change (they are, for example, spatial regions tied to the surface of the Earth). Consider again the examples in Figure 4. The granular partition projecting onto the United States has existed for more than one hundred years without significant changes, where each area of bad weather changes continuously throughout the course of its (brief) existence. In fact Figure 4 itself needs to be considered as a snapshot (Smith and Brogaard, to appear). Due to the relative stability of the reference partition in a case such as this, it provides useful information to say that the area of bad weather was located in parts of Montana, Idaho, Wyoming, and Utah at such and such a time. Every child learns this reference partition at school, and it is used for all sorts of purposes thereafter (Stevens and Coupe 1978).

(2) means that the skeleton of a reference partition must be mereologically monotonic in the sense of (Bittner and Smith 2001). This is in order to assure that the mereological structure of the underlying domain is preserved. This means that if objects recognized by a given partition stand to each other in a relation of part to whole, then the cells in which these objects are located stand to each other in the subcell relation.

Conditions (2)-(5) ensure that each crisp reference partitions is full, exhaustive, and complete in the sense of (Smith and Bittner 2001) and that it creates a subdivision of the targeted domain into jointly exhaustive and pairwise disjoint portions. Important groups of reference partitions are partitions imposed by quantities of all kinds (Johansson 1989, chapter 4); temporal partitions like calendars (Bittner to appear), and spatial partitions like political or administrative subdivisions.

8. CONCLUSIONS

We have proposed an application of the theory of granular partitions to the phenomenon of vagueness, a phenomenon which is itself seen as a semantic property of names and predicates. We defended a supervaluationistic theory of the underlying semantics but in contrast to standard supervaluationism we argued that vague names must be evaluated semantically as they appear within the judgments actually made in natural contexts. We showed that the use of frames of reference in making approximating judgments can be formulated very naturally in partition-theoretic terms, and that the framework of granular partitions then helps us to understand the relationships between vagueness and approximation.

ACKNOWLEDGEMENTS

The authors thank Wolfgang Heydrich for helpful comments. This work was supported by DARPA under the Command Post of the Future program, by the National Science Foundation under the Research on Learning and Education program, and under Research Grant BCS-9975557: "Geographic Categories: An Ontological Investigation".

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