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Vague Reference and Approximating Judgments

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'Mount Everest' is a vague name. That is (on the account here defended) there are many portions of reality all of which have equal claims to serve as its referent. We propose a new account of such vagueness in terms of a theory of what we shall call *granular partitions*. We distinguish different kinds of crisp and non-crisp granular partitions and we describe the relations between them, concentrating especially on spatial examples. In addition, we describe the practice whereby subjects use systems of reference grids as a means for tempering the vagueness of their judgments, for example when they say that Libya straddles the Equator or that the meeting will take place between 2 and 3pm. We then demonstrate how the theory of reference partitions can yield a natural account of this practice, which is referred to in the literature as 'approximation'.

Keywords: ontology, granular partitions, vagueness, semantic partitions, partition theory, approximation

Consider the proper name 'Mount Everest'. This refers to a mereological whole, a certain giant formation of rock. A mereological whole is the sum of its parts, and Mount Everest certainly contains its *summit* as part. But it is not so clear which parts along the foothills of Mount Everest are parts of the mountain and which belong to its surroundings. Thus it is not clear which mereological sum of parts of reality actually constitutes Mount Everest. One option is to hold that there are multiple candidates, no one of which can claim exclusive rights to serve as the referent of this name. All of these candidates are involved, in some

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sense, when we use the name 'Mount Everest.' We are however not conscious of this multiplicity of candidate referents, effectively because we simply do not care about the question where, precisely, the boundaries around Mount Everest are to be drawn.

Each of the many candidates has the summit, with its height of 29,028 feet, as part. Each is also a perfectly determinate portion of reality. The candidates differ only regarding which parts along the foothills are included and which are not.

Varzi (2001) refers to the above as a *de dicto* view of vagueness. It treats vagueness not as a property of objects but rather as a semantic property of names and predicates, a property captured formally in terms of a supervaluationistic semantics (Fraassen 1966), (Fine 1975). We shall concentrate our attentions in what follows on the case of *singular reference*, i.e., reference via names and definite descriptions to concrete portions of reality such as mountains and deserts. We shall also concentrate primarily on spatial examples. As will become clear, however, it is one advantage of the framework here defended that it can be generalized automatically beyond the spatial case.

In order to understand vague reference we use the theory of granular partitions we advanced in our earlier papers: (Bittner and Smith 2001a), (Bittner and Smith 2003), (Smith and Brogaard 2002). The fundamental idea is that every use of language to make a judgment about reality brings about a certain *granular partition*, a grid-like system of cells conceived as projecting onto reality in something like the way in which a bank of flashlights projects onto reality when it carves out cones of light in the darkness. Each judgment, J, can then be conceived as a pair consisting of a sentence, S, and an associated granular partition Pt.

We consider reference as a *two-step-process*. Language tokens are associated with cells in a grid-like structure, and these cells are projected onto reality in the way suggested by our flashlight metaphor. Granular partitions can then be conceived as the cognitive artifacts whereby language gains its foothold in reality. (They thus play a role somewhat similar to that of set-theoretical models in more standard treatments.) In our earlier papers, we showed how this two-step-process allows us to explain the features of selectivity and granularity of reference in judgments. In this paper, we show how the same machinery can help us to understand the phenomena of vagueness and approximation.

Crisp Granular Partitions

The theory of granular partitions has two parts: (A) a theory of the relations between cells and the structures they form, and (B) a theory of the relations between cells and objects in reality. Consider Figure 1. The left part shows a very simple cell structure, with cells labeled *Everest*, *Lhotse* and *The Himalayas*. The right part shows portions of reality onto which those cells project.

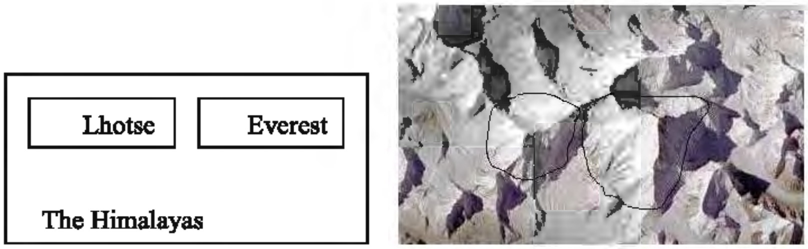

Figure 1: Left: a partition, with cells *Lhotse*, *Everest* and *The Himalayas*. Right: A part of the Himalayas seen from space, with admissible candidate referents for 'Mount Lhotse' (left) and 'Mount Everest' (right).

Language

In what follows, we use lower case roman letters o, o_1, o_2, … to symbolize objects in reality; z, z_1, z_2, to symbolize cells of granular partitions; upper case roman letters from the beginning of the alphabet A, B, C, … to symbolize sets of cells; upper case roman letters from the middle of the alphabet L, P, … to symbolize sets of ordered tuples; and upper case Greek letters Δ, Δ_1, ... to symbolize sets of objects in reality.

Theory A

A granular partition $Pt = ((A, \subseteq), (\Delta, \leq), P, L)$ is a quadtuple such that $(A, \subseteq)$ is a system of cells or a *cell-structure*, $(\Delta, \leq)$ is a *target domain*, $L \in Pow(\Delta \times A)$ is a *location* relation, and $P \in Pow(A \times \Delta)$ is a *projection* relation. The target domain $(\Delta, \leq)$, is hereby understood as a mereological structure with Δ a set of objects and $\leq$ a part-of relation defined on Δ which satisfies the axioms of general extensional mereology (GEM). A cell structure, $(A, \subseteq)$, is a finite set of cells, z_0, z_1, … z_n with a binary *subcell* relation $\subseteq$. We say that z_1 is a subcell of z_2 in A if and only if the first is contained in the latter. We then impose four axioms (or 'master conditions') on cell structures as follows:

> MA1: The subcell relation $\subseteq$ is reflexive, transitive, and antisymmetric.
> MA2: The cell structure of a partition is always such that chains of nested cells are of finite length.
> MA3: If two cells have subcells in common, then one is a subcell of the other.
> MA4: Each partition contains a unique maximal cell.

These conditions, which are explored further in our earlier papers, together ensure that each cell structure can be represented as a tree (a directed graph with a root and no cycles).

Theory B

Theory (B) arises in reflection of the fact that partitions are more than just systems of cells. They are constructed in such a way as to project upon reality in the way names and other referring expressions in natural and scientific languages project onto entities in reality. Projection and location then are relations between cells in a cell structure on the one hand and objects in a target domain on the other. We write 'P(z, o)' as an abbreviation for: cell z is projected onto object o, and 'L(o, z)' as an abbreviation for: object o is located in cell z. The partitions of interest in this paper are *transparent*, which means that MB1 and MB2 hold:

MB1 $L(o, z) \rightarrow P(z, o)$.
MB2 $P(z, o) \rightarrow L(o, z)$.

(Here and in what follows initial universal quantifiers are taken as understood. We preserve L and P as distinct relations in order to hold open the possibility of dealing with certain sorts of breakdown in the relation between granular partitions and their targets.)

We demand further that projection and location be functional relations, i.e., that every cell projects onto just one object and every object is located in just one cell:

MB3 $P(z, o_1)$ and $P(z, o_2) \rightarrow o_1 = o_2$
MB4 $L(o, z_1)$ and $L(o, z_2) \rightarrow z_1 = z_2$

The partitions of interest in this paper are in addition *complete*, in the sense that every cell projects onto at least one object, i.e., they satisfy an axiom to the effect that they contain no empty cells (no cells projecting outwards into the void):

MB5 $z \in A \rightarrow \exists o\colon L(o, z)$

We require also that projection, considered as a function p: $A \rightarrow \Delta$ between two partially ordered domains (A and Δ), be an order homomorphism:

MB6: $z_1 \subseteq z_2 \rightarrow p(z_1) \leq p(z_2)$

The root or maximal cell in the cell structure is then the maximal object (the universal or total fusion) in Δ.

The resulting class of partitions is quite narrow. For a more general treatment, embracing also less well-behaved granular partitions, see (Bittner and Smith 2003). Note also that our axioms MB1-6 have been formulated for easy understandability and the system they form is not minimal. (Thus MB2 already follows from MB1, MB3 and MB5.) In order to simplify the notation in what follows, we write $Pt = (A, P, L)$ as an abbreviation for $Pt = ((A, \subseteq), (\Delta, \leq), P, L)$. At the same time we assume a fixed target domain Δ, which the reader can think of as the whole of reality.

Vague Granular Partitions

The Theory

What, now, of vagueness? A *vague* granular partition $Pt^V = ((A, \subseteq), (\Delta, \leq), P^V, L^V)$ is a quadtuple in which the cell structure and target domain are defined as above, and P^V and L^V are *classes* of projection and location relations (Bittner and Smith 2001b). Again, we will write $Pt^V = (A, P^V, L^V)$ in order to keep the notation simple.

Consider Figure 2, which depicts a vague partition $Pt^V = (A, P^V, L^V)$ of the Himalayas. This has a cell structure A, as shown in the left part of Figure 2, which is in fact identical to the corresponding part of Figure 1. In the right part of the figure, in contrast, there is a multiplicity of possible candidate projections for the cells in A, indicated by boundary regions depicted via cloudy ovoids. The boundaries of the actual candidates onto which the cells 'Lhotse' and 'Everest' are projected under the various P_i in P^V are continuous ovoids included somewhere within the cloudy regions depicted in the Figure.

The projection and location relations in these classes form pairs (P_i, L_j), which are such that each P_i has a corresponding unique L_j and vice versa, satisfying the following conditions (where the notation '$\exists!i$' abbreviates: 'there exists one and only one i'):

$MB1^V$ $\forall j: L_j(o, z) \rightarrow \exists! i\ P_i(z, o)$
$MB2^V$ $\forall i: P_i(z, o) \rightarrow \exists! j: L_j(o, z)$

We also demand that all P_i and all L_j are functional in the sense discussed in the crisp case:

$MB3^V$ $P_i(z, o_1)$ and $P_i(z, o_2) \rightarrow o_1 = o_2$
$MB4^V$ $L_j(o, z_1)$ and $L_j(o, z_2) \rightarrow z_1 = z_2$

Figure 2: A vague partition of the Himalayas

We demand further that cells project onto some object (are non-empty) under every projection:

$$\text{MB5}^V\text{:}\quad Z(z, A) \rightarrow \forall j\, \exists o\text{: } L_j(o, z)$$

Again, every particular projection considered as a function from A to Δ is an order homomorphism:

$$\text{MB6}^V\text{:}\quad z_1 \subseteq z_2 \rightarrow p_i(z_1) \leq p_i(z_2)$$

We now add an axiom that governs the interrelations between projection relations with distinct indexes in the vague partition Pt^V. Recall that all projection relations operate on the same cell structure. We need to ensure that if the same object is targeted by two cells z_1 and z_2 under different projections P_i and P_j then the targeting cells must be identical:

$$\text{MB7}^V\quad P_i(z_1, o) \text{ and } P_j(z_2, o) \rightarrow z_1 = z_2$$

Equivalence of Candidate Referents

Given a vague partition as defined above, we can define an equivalence relation between entities in the target domain Δ as

$$\text{D}{\approx}\qquad o_1 \approx o_2 \equiv \exists z, i, j : P_i(z, o_1) \text{ and } P_j(z, o_2).$$

Clearly, ≈ is symmetric and reflexive. To see that ≈ is also transitive, assume $o_1 \approx o_2$ and $o_2 \approx o_3$. This means that there exist z_1, z_2, i, j, k, m such that $P_i(z_1, o_1)$, $P_j(z_1, o_2)$, $P_k(z_2, o_2)$ and $P_m(z_2, o_3)$. From $P_j(z_1, o_2)$ and $P_k(z_2, o_2)$ it follows by MB7^V that $z_1 = z_2$, and hence for some cells z and some projections P_i and P_m it holds that $P_i(z, o_1)$ and $P_m(z, o_3)$, i.e., $o_1 \approx o_3$. In the remainder, we write $[o]_z$ to denote the set $\{o \mid \exists\, i : P_i(z, o)\}$.

Let $Pt^V = (A, P^V, L^V)$ be a vague granular partition. We call all partitions $Pt = (A, P_i, L_j)$ with $P_i \in P^V$ and $L_j \in L^V$ which satisfy the axioms MB1–MB6 *crispings* of the vague partition Pt^V. Consider a partition with cells labeled with vague proper names. Intuitively, each crisping (A, P_i, L_j) then recognizes exactly one candidate precisified referent for each such cell. The precise candidates carved out by the separate (A, P_i, L_j) are all slightly different. But each is perfectly crisp and thus it has all of the properties of crisp partitions discussed in the previous sections. All those different candidate referents are equivalent in the sense of our relation ≈. This captures the *de dicto* view of vagueness.

Semantic partition

Given a vague partition $Pt^V=((A^V, \subseteq), (\Delta, \leq), P^V, L^V)$, we can for each cell $z \in A^V$ classify corresponding portions of reality with respect to the vague projection of the cell z into three zones: the determinate zone, the indeterminate zone, and the exterior zone.

We say that x is part of the *determinate core* of the vague projection P^V of the cell z if and only if, under *all* projections $p_i(z)$ in P^V, x is a part of the targeted candidate referent:

$$\text{determinate}_V(x, z) \equiv \forall i\text{: } x \leq p_i(z)$$

Thus the summit of Mount Everest is part of the determinate core of the vague projection of the name 'Everest' and its associated cell.

We say that x is a part of the *indeterminate* zone of the vague projection P^V of the cell z if and only if there are some projections p_i in P^V under which x is part of the targeted candidate referent and other projections $p_j(z)$ in P^V under which this is not the case:

$$\text{indeterminate}_V(x, z) \equiv \exists i\text{: } x \leq p_i(z) \text{ and } \exists j\text{: } \neg(x \leq p_j(z))$$

The dotted region in Figure 2 illustrates the indeterminate zone of the projection of the cell associated with the vague name 'Mount Everest'.

We say that x is a part of the zone *exterior* to the vague projection P^V of the cell z if and only if x is not reached by any projection $p_i(z)$ in P^V.

$$\text{exterior}_V(x, z) \equiv \forall i\text{: } \neg(x \leq p_i(z))$$

There are parts of reality – such as Berlin – that are not reached by any projections of the cell 'Everest' in the partitions used by humans projecting in transparent fashion.

Reference via a vague name 'N' creates a partition of reality into determinate core, indeterminate zone, and exterior zone. Let $\sigma x \psi(x)$ denote the mereological sum of all x satisfying $\psi(x)$ and let z be the cell in the partition P^V which projects onto the candidate referents for 'N'. We then define determinate core, indeterminate zone, and exterior zone as the mereological sums of all determinate, indeterminate, and exterior parts of reality with respect to the vague projection of the cell z:

$$\mathit{det}_V(z) = \sigma x \text{ determinate}_V(x, z)$$
$$\mathit{indet}_V(z) = \sigma x \text{ indeterminate}_V(x, z)$$
$$\mathit{ext}_V(z) = \sigma x \text{ exterior}_V(x, z)$$

We define the *semantic partition* of reality with respect to the vague name N as a triple of determinate zone, indeterminate zone, and exterior zone. In general $\mathit{det}_V(z)$ is a partial function since there does not necessarily exist a portion of reality which is a part of all projections of the cell z.

Approximating Judgments

Approximation in Egg-yolk Partitions

Consider, again, Figure 1. The cells labeled 'Everest' and 'Lhotse' carve *mountain-candidates* out of a certain formation of rock. They do not do this physically, but rather by establishing fiat boundaries in reality, represented by the black lines in the right part of the figure (Smith 1995), (Smith 2001), (Bittner and Smith 2001a). But how are we to understand the phenomenon whereby judging subjects are able to impose spatial boundaries *vaguely*?

Suppose you are an expert mountain guide hiking through the Himalayas with your friends and you assert:

[A]: We will cross the boundary of Mount Everest within the next hour.

We shall assume that through your use of the phrase 'within the next hour' you successfully delimit a range of admissible candidates for the boundary of Mount Everest along the trajectory of your hike. Consider the left part of Figure 3. Here boundaries delimiting admissible candidates are imposed by specifying a time interval that translates to travel distance along a path; time serves here as *frame of reference*. The boundaries are defined by your current location (marked: 'now') and your location after the specified time has passed (marked: 'in one hour'). The boundary of each admissible candidate referent crosses the path at some point between these two boundaries, called the *exterior* and the *interior* boundaries, respectively.

The general case is illustrated in the right part of Figure 3, which is intended to depict how judging subjects project egg-yolk-like granular partitions onto reality involving three cells: an exterior, a core, and an intermediate region within which the boundary candidates lie. (See (Cohn and Gotts 1996) and (Roy and Stell 2001).) This granular partition serves as the frame of reference in terms of which the judging subject is able at the same time to both specify the range of admissible entities to which he (vaguely) refers and also to constrain this range.

Egg-yolk Partitions vs. Semantic partition

Consider the egg-yolk partition in the right part of Figure 3. It is important to see that this is not a semantic partition in the sense discussed above. This is because it was created by a judging subject by imposing boundaries onto reality in order to constrain admissible candidate referents and at the same time to serve as a frame of reference.

A semantic partition on the other hand is induced by classifying portions of reality into determinate zone, indeterminate zone, and exterior zone with respect to the vague projection of a certain cell in a vague partition. Ideally, when corresponding to the same vague name, both partitions coincide but, as we

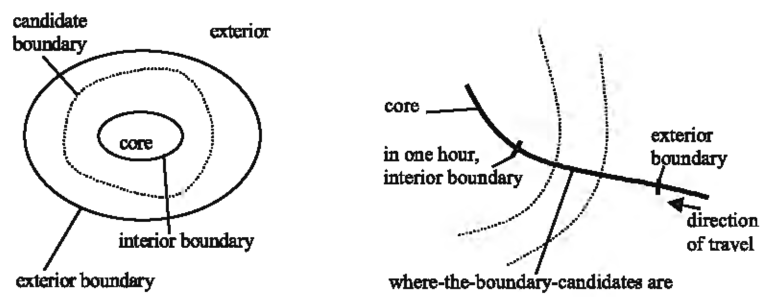

Figure 3: Egg-yolk like reference partitions

shall see below, this is not the case in general. We will discuss the relationships between the two kinds of partitions in a later section. Until then we ignore the notion of semantic partition and focus on the kinds of partitions shown in the right part of Figure 3 and their use as frames of reference in approximations.

Approximation in Complex Partitions

In the case just discussed, new fiat boundaries are created by judging subjects in ad hoc fashion in order to delimit vagueness. But there are also cases where already existing systems of boundaries are *re-used* for this same purpose. There is one crisp granular partition of this sort with which we are all familiar. It has exactly 50 cells, which project onto the 50 United States of America. A fragment of this partition is presented in the left and right parts of Figure 4. In the foreground of the figure we see in addition an area of bad weather (also called 'Hurricane Walter'), represented by a dark dotted region that is subject to vagueness *de dicto* in the sense discussed above. Wherever the boundaries of this object might be located, they certainly lie skew to the boundaries of the relevant states. But the figure also indicates (with the help of suitable labeling) that:

> [B] Hurricane Walter extends over parts of Wyoming, parts of Montana, parts of Utah, and parts of Idaho.

In the sorts of contexts which we humans normally inhabit, it is impossible to refer to any *crisp* boundary when making judgments about the location of a region of bad weather of the sort described. However, it is possible to describe its (current) location relative to the grid of a map in the manner illustrated in judgment [B].

We, the judging subjects, then deliberately employ a corresponding partition as our frame of reference and we describe the *relationships* that hold between all admissible referents of the vague term 'Hurricane Walter' and the cells of this partition. In terms of spatial relations, this means in the given case that all admissible candidates partially overlap the states of Wyoming, Montana, Utah, and Idaho and that they do not overlap any other state. Consequently, if a judging subject can specify for every partition cell a unique relation – for example *part of* – that holds for all admissible candidate referents of a vague term, then this is a ***determinate way to effect vague reference***. The technical name for this phenomenon is ***approximation***. For details, see (Bittner and Stell 2002).

A meteorologist may achieve a finer approximation by employing a finer-grained partition as frame of reference in order to make a more specific judgment about the current location of the bad weather region. Thus she might use cells labeled Eastern Idaho, Southern Montana, Western Wyoming, and Northern Utah, and so on, yielding a fiat boundary of the sort depicted in the right part of Figure 4.

Notice that all these boundaries predate the judgments which use them as frames of reference in relation to this particular bad weather system. They are there to be used over and over again in formulating constraints on the possible locations of admissible candidate referents corresponding to vague referring terms. They represent a convenient and determinate way to make vague reference, which has even greater utility when the frame of reference is a commonly accepted one, as in the present case.

Approximation and Judgments

Approximating judgments are a special class of judgments that contain both vague names and a (relatively) crisp reference to boundaries that delimit this vagueness. [A], too, is an approximating judgment which contains the vague name 'Everest' and also a reference to boundaries delimiting the vagueness of

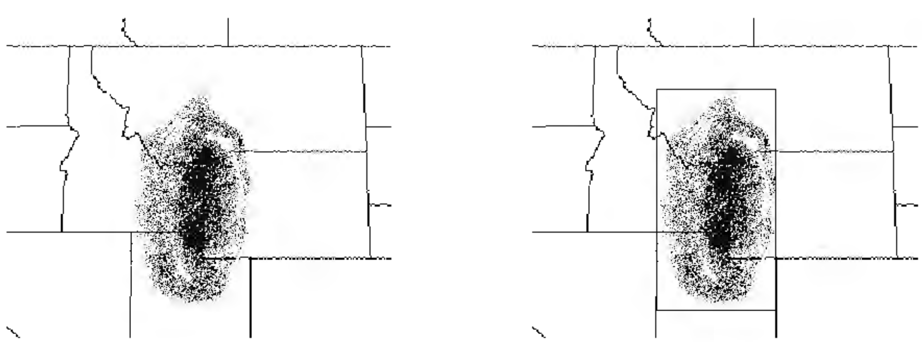

Figure 4: States of the United States with Hurricane Walter

this term via the phrase '[crossable] within the next hour'. In this paper, we consider approximating judgments which contain a single vague name and a crisp reference frame. More complex cases are possible – including the case where the reference frame itself involves a certain degree of vagueness – but formal consideration of the latter is omitted here since its treatment follows the same basic pattern.

An approximating judgment J^A, if uttered successfully, imposes *two* partitions onto reality: a *vague* partition Pt^V and a *reference partition* Pt^R, along the lines above, whereby the latter serves to delimit the vagueness of the former. An approximating judgment J^A is thus a triple (S, Pt^V, Pt^R), consisting of a sentence, S, together with two granular partitions, Pt^V and Pt^R.

In the approximating judgment $J^A = ([B], Pt^V, Pt^R)$, expressed by the sentence: 'Hurricane Walter extends over parts of Wyoming, Montana, Utah, and Idaho', the corresponding vague partition Pt^V contains a cell labeled 'Hurricane Walter', which projects onto a multiplicity of admissible candidates. At the same time this judgment reuses the partition depicted in the left part of Figure 4 as its reference partition Pt^R. The latter constrains the admissible projections of the cell labeled 'Hurricane Walter' in Pt^V in such a way that each candidate referent that is targeted by a projection P_i of Pt^V must extend over parts of reality targeted by the cells of Pt^R labeled 'Wyoming', 'Utah', 'Montana', and 'Idaho' respectively.

Partition Theory and Approximation

The idea underlying the partition-theoretic view of approximation is that a (crisp) granular partition can be used as a frame of reference (a generalized coordinate frame (Bittner 1997)), which allows us (a) to describe the *approximate location* of objects and thus (b) to project onto portions of reality in an approximate way. We call a granular partition which is used as a frame of reference in this manner a *reference partition.*

Consider a vague name such as 'Hurricane Walter' (hereafter: 'HW') and the corresponding multiplicity of admissible candidate referents for this name formed by crisp portions of reality in the domain of the northwestern United States at some given point in time. Consider some crisp partition structuring this same domain but without recognizing any of the candidates referred to by the name 'HW' directly. This might be the partition created by the boundaries of the separate States of the sort used in Figure 4, or it might be a partition formed by a raster of cells aligned to lines of latitude and longitude.

To understand the formal details of how the latter can serve as reference partition in relation to the former we introduce the three concepts of full overlap *(fo)*, partial overlap *(po)*, and non-overlap *(no)*, concepts which we shall now use to generalize the notions of projection and location, as follows. Consider a reference partition whose cells are projected onto regions of space on the surface of the Earth. Let o be a portion of reality that straddles the boundaries of the cells of this reference partition. The constants *fo*, *po*, *no* will now be used to measure the degree of mereological coverage of the object o by the corresponding regions of space.

We call the relation $L^R(o, z, \omega)$ the *rough location* of the portion of reality o with respect to the cell z and the relation $P^R(z, o, \omega)$ the *rough projection* of the cell z onto o. (We use the phrases 'rough location' and 'rough projection' in order to emphasize our indebtedness to the account of approximation in terms of rough sets advanced in (Pawlak 1982).) In both relations, ω stands for the degree of mereological overlap of the portion of reality targeted by the cell z with the actual portion of reality o, i.e., it takes one or other of the values *fo*, *po*, or *no*. Consider the left part of Figure 4. There the relation *po* holds between all admissible candidate referents HW_i and Montana, i.e., $\forall i$: $L^R(HW_i$, Montana, *po*). The relation *no* holds between all the HW_i and Oregon, i.e., $\forall i$: $L^R(HW_i$, Oregon, *no*).

We can characterize the relationships between exact and rough location and exact and rough projection in reference partitions as follows:

$$
\begin{aligned}
L^R(o, z, fo) &\equiv \exists x\ (L(x, z) \text{ and } x \leq o)\\
P^R(z, o, fo) &\equiv \exists x\ (P(z, x) \text{ and } x \leq o)\\
L^R(o, z, po) &\equiv \exists x(\ L(x, z) \text{ and } \exists y\ (y \leq x \text{ and } y \leq o) \text{ and}\\
&\qquad \exists y\ (y \leq x \text{ and } \neg(y \leq o)))\\
P^R(z, o, po) &\equiv \exists x\ (P(z, x) \text{ and } \exists y\ (y \leq x \text{ and } y \leq o) \text{ and}\\
&\qquad \exists y\ (y \leq x \text{ and } \neg(y \leq o)))\\
L^R(o, z, no) &\equiv \exists x\ (L(x, z) \text{ and } \neg\exists y\text{: } y \leq x \text{ and } y \leq o)\\
P^R(z, o, no) &\equiv \exists x\ (P(z, z) \text{ and } \neg\exists y\text{: } y \leq x \text{ and } y \leq o)
\end{aligned}
$$

The notion of rough location gives rise to an equivalence relation in the domain of objects (portions of reality), with respect to a given reference partition Pt^R with rough location relation L^R, as follows:

$$\text{D}\sim \qquad o_1 \sim o_2 \equiv \forall z, \omega\text{: } L^R(o_1, z, \omega) \leftrightarrow L^R(o_2, z, \omega).$$

Thus two objects are equivalent with respect to the granular partition Pt^R if and only if they have an identical rough location with respect to all cells of this partition. The relation ~ can thus be interpreted as meaning: indiscernibility with respect to the frame of reference provided by Pt^R. In an approximating judgment $J^A = (S, Pt^V, Pt^R)$, the reference partition Pt^R will be chosen in such a way that the candidate referents targeted by a single cell in Pt^V are equivalent with respect to ~. Often there may be a number of possible choices for reference partitions, all of which have the feature that all candidate referents of the vague name in question are equivalent with respect to the indiscernibility relation ~ induced by the reference partition. For example in Figure 4 we could also have used a regular (raster-shaped) reference partition of some appropriate resolution. However, it is more appropriate in a weather forecast to use the reference partition defined by the boundaries of the separate States because of its familiarity. We will discuss different choices of reference partitions in later sections.

We define a *reference partition* as a quintuple, $Pt^R = ((A, \subseteq), (\Delta, \leq), P^R, L^R, \Omega)$ where $(A, \subseteq)$ and $(\Delta, \leq)$ are a cell structure and target domain as specified above, P^R and L^R are rough projection and location relations, and Ω is the set of values (*fo*, *po*, *no*) indicating degrees of overlap (coarser and finer distinctions are possible, as discussed in (Bittner and Stell 2003)). We then can prove that the following counterparts of MB1-3 hold for reference partitions:

TR1 $L^R(o, z, \omega) \rightarrow P^R(z, o, \omega)$

TR2 $P^R(z, o, \omega) \rightarrow L^R(o, z, \omega)$

TR3 $(\forall z, \omega: P^R(z, o_1, \omega) \leftrightarrow P^R(z, o_2, \omega)) \rightarrow o_1 \sim o_2$

TR1 follows from MB1. To see this assume $L^R(o, z, \omega)$ and let $\omega = fo$. We have $\exists x$ (L(x, z) and $x \leq o$). By MB1 we have $\exists x$ (P(z, x) and $x \leq o$), hence $P^R(z, o, fo)$ and similarly for $\omega = po$ and $\omega = no$. TR2 follows from MB2 in a similar manner. To see TR3 assume $\forall z,\omega$ $(P^R(z, o_1, \omega) \leftrightarrow P^R(z, o_2, \omega))$. By TR1 and TR2 L^R and P^R are logically equivalent and can be substituted for each other. Therefore we have $L^R(o_1, z, \omega) \leftrightarrow L^R(o_2, z, \omega)$, i.e., $o_1 \sim o_2$.

Corresponding to MB4 we now demand that if all objects have the same relation $\omega \in \{fo, po, no\}$ with respect to the cells z_1 and z_2 then these two cells are identical:

R1 $(\forall o, \omega: L^R(o, z_1, \omega) \leftrightarrow L^R(o, z_2, \omega)) \rightarrow z_1 = z_2$.

Constraining Approximation

Well-Formed Approximations

If an approximating judgment like ([A], Pt^V, Pt^R) is to succeed, that is if a true judgment of this form is to have been made, then the reference partition needs to project onto reality in such a way that all admissible candidate referents are equivalent with respect to the indiscernibility relation imposed by Pt^R. Thus, in the hiker case, each value of p^V_i('Everest') must be such that its boundary can be crossed in one hour from the time when the judgment is made (Figure 3).

Let (S, Pt^V, Pt^R) be an approximating judgment and let Pt^V be a vague partition with a cell for each vague name in the sentence S. We then demand that in such an approximating judgment the reference partition Pt^R and the vague partition Pt^V be related to each other in such a way that candidate referents which are targeted by the same cell (i.e., are equivalent in the sense of $\approx$) have the same rough approximation in the underlying reference partition (i.e., are also equivalent in the sense of $\sim$):

EP $o_1 \approx o_2 \rightarrow o_1 \sim o_2$.

We call EP the *equivalence principle*. EP rules out many reference partitions Pt^R which cannot be used for constraining the vagueness of the vague partition Pt^V. Thus it rules out, for example, reference partitions with resolutions too fine for the degree of vagueness of the corresponding vague partition. Consider Figure 4. A raster-cell-partition with cell size of $1m^2$ would violate the equivalence principle, since not all candidate referents of the vague name 'Hurricane Walter' would be indiscernible with respect to this reference partition. If the reference partition is too fine then equivalence in the sense of ≈ does not imply equivalence in the sense of ~.

Notice that the converse of the equivalence principle does not hold. This is because there might be portions of reality (x and y) that are equivalent with respect to the reference partition (x ~ y), but which are such that neither is a candidate referent targeted by the cell in question. Consider Figure 5. The approximation of the mereological sum of Yellowstone National Park and Zion National Park (YNP + ZNP) with respect to the Federal State reference partition (left) is identical to the approximation of the candidate referents of the name 'Hurricane Walter' (right) but surely (YNP + ZNP) is not a candidate referent for the name 'Hurricane Walter'.

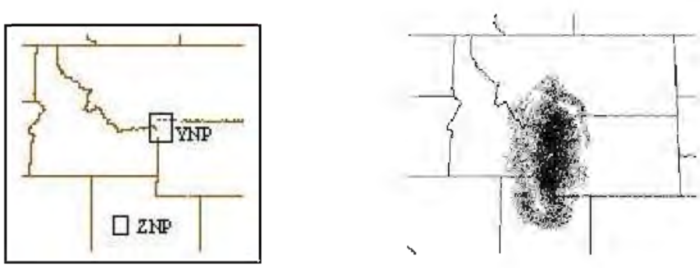

Figure 5 Left: Yellowstone National Park (YNP) and Zion National Park (ZNP). Right: Hurricane Walter.

Precise Approximation

In this section, we discuss the relationship between semantic partition and the kinds of reference partitions previously discussed.

Consider the approximating judgment J = ([A], Pt^V, Pt^R). The reference partition Pt^R shown in the left part of Figure 3 imposes two fiat boundaries onto reality: an interior boundary of the approximation and an exterior boundary of the approximation. As discussed above, this often results in a partition structure similar to the one depicted in the right part of the figure. The projection of this partition onto the path the judging subject takes on her journey towards the summit of Mount Everest results in the reference partition Pt^R.

Consider now the semantic partition imposed by the cell labeled 'Everest' in the vague partition Pt^V. We can see that the relationship between Pt^V and Pt^R

satisfies the equivalence principle EP, which demands that all candidate referents of the vague name 'Everest' are equivalent under ~. Consider now the location of two pairs of boundaries: (a) the interior and exterior boundaries imposed by the judging subject as a frame of reference for her approximation; and (b) the boundaries imposed by the semantic partition into determinate zone, indeterminate zone, and exterior zone via the cells of Pt^V. We say that the approximating judgment is *precise* if and only if (1) the interior boundary of the approximation coincides with the boundary separating the determinate zone from the surrounding parts of the semantic partition; and (2) the exterior boundary of the approximation coincides with the boundary separating the exterior zone from the indeterminate zone of the semantic partition. This means that the semantic partition and the reference partition coincide.

In order to take more complex reference partitions into account, we now define upper and lower approximations of an object o with respect to such partitions. (Again, we use the notions of lower and upper approximation in order to emphasize the correspondence to rough set theory of Pawlak (1982).) The lower approximation of an object o with respect to a reference partition Pt^R is the mereological sum of all those portions of reality which are targeted by cells of Pt^R and which are contained in o:

$$\text{Lower}(o) = \sigma o'(\exists z(o' = p(z) \mathbin{\&} P^R(z, o, \textit{fo}))),$$

where $\sigma x\phi(x)$ is defined as above.

The upper approximation is the mereological sum of all those portions of reality which are targeted by cells of the reference partition and which overlap o:

$$\text{Upper}(o) = \sigma o'(\exists z(o' = p(z) \mathbin{\&} (P^R(z, o, \textit{fo}) \text{ or } P^R(z, o, \textit{po}))))$$

Consider now the reference partitions shown in the left part of Figure 6, which is a refined version of the egg-yolk reference partition in the right part of Figure 6. The core cell of the latter is subdivided into eastern and western subcells (ec and wc for eastern part of the core region and western part of the core region, respectively). Moreover the region where the boundaries are is

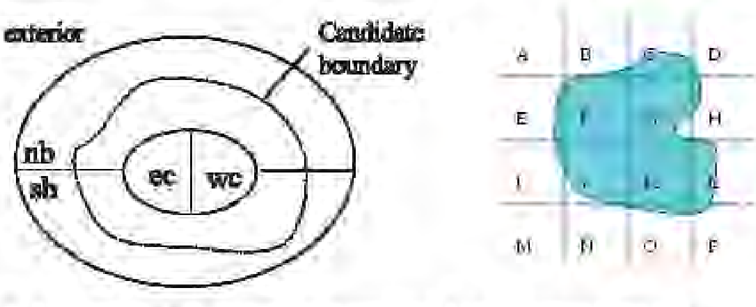

Figure 6: Approximation in complex partitions. Left: a refined egg-yolk partition. Right: a raster partition.

subdivided into a northern and southern region (nb and sb, respectively). The lower approximation of the candidate referent signified by its outer boundary is the mereological sum of those portions of reality which are targeted by the cells 'ec' and 'wc'. Its upper approximation has as parts in addition portions of reality targeted by the cells 'nb' and 'sb'.

For another example of lower and upper approximations, consider the right part of Figure 6. The upper approximation of the depicted region with respect to the raster-shaped partition is the mereological sum of the targets of the cells [B,…, L, O, P]. The lower approximation is identical to the target of the cell K.

Notice that upper approximations are always defined. Lower approximations, however, are only defined if the reference partition is sufficiently fine grained, so that $P^R(z, o, fo)$ holds for some cell z and some portion of reality o. This is because in mereology there is no counterpart to the empty set.

With respect to these more complex reference partitions we now say that an approximating judgment is precise if and only if (1) the boundary of the lower approximation of any candidate referent of 'N' coincides with the boundary separating the determinate zone from the surrounding parts of the semantic partition imposed by the vague projection of the cell associated with 'N'; and (2) the boundary of the upper approximation of any of the candidate referents coincides with the boundary separating the exterior zone from the indeterminate zone of the vague projection of the cell associated with 'N'.

In formal terms, we describe this as follows. Let $J = (N, Pt^V, Pt^R)$ be an approximating judgment with $Pt^V = ((A^V, \subseteq), (\Delta, \leq), P^V, L^V)$ and $Pt^R = ((A^R, \subseteq), (\Delta, \leq), P^R, L^R, \Omega)$. We then call J^A *precise* if and only if

$$\forall o \in [o]_z\text{: Lower}(o) = det^V(z) \text{ and Upper}(o) = (det^V(z) + indet^V(z)).$$

Here $z \in A^V$ is the cell in the vague partition corresponding to the vague name 'N', $[o]_z$ is the set of all objects targeted by the vague projection of z, and + is the mereological sum.

Constraining Approximation

We now discuss constraining approximations, defined as those approximations that do not have the property of being precise but still satisfy the equivalence principle.

Let z be the cell in the vague partition Pt^V which corresponds to the vague name 'N', and let $[o]_z$ be the set of all candidate referents of 'N', i.e., portions of reality targeted by z under P^V. The approximation of candidate referents $o \in [o]_z$ with respect to the approximating partition Pt^R is called *constraining* if and only if the following holds:

$\forall o \in [o]_z$:

either:

Lower(o) is defined and Lower(o) $\leq det_V(z) \leq (det_V(z) + indet_V(z)) \leq$ Upper(o)

or:

$det_V(z) \leq (det_V(z) + indet_V(z)) \leq$ Upper(o)

Consider Figure 4 and assume that the determinate zone of the vague reference of the name 'Hurricane Walter', det_V(HW), is situated along the border between Idaho and Wyoming. It follows that the lower approximation is undefined for any of the candidate referents because the federal state partition is too coarse. The upper approximation however *is* defined, since any candidate referent is part of the mereological sum of Idaho, Montana, Wyoming, and Utah. Hence the resulting approximation is constraining.

Consider the class of constraining approximations. As already Aristotle repeatedly emphasized (at 1094b11 *sq.*, 1098a26, 1103b34 *sq.*, 1165a13), judging subjects will characteristically use those approximating judgments which are constraining but which are *as precise as necessary* in whatever is the context in hand. In (Bittner and Smith 2001b), we argue that this will imply that the limits imposed on vagueness by an approximation will normally be such that the resulting judgment is not subject to truth-value indeterminacy. The judgments we actually make in normal contexts (as contrasted with those types of artificial judgments invented by philosophers) are determinately either true or false even in spite of the vague terms which they contain.

Properties of Reference Partitions

Reference partitions are of central importance for approximating judgments. Examples of reference partitions include: any political subdivision, raster-shaped partitions adjusted to latitude and longitude, the block structure in American cities, the subdivision of Vienna into *Bezirke* and of France into *Départements*, etc. Other important groups of reference partitions are partitions imposed by quantity-scales of all kinds (Johansson 1989, chapter 4), including temporal partitions like calendars (Bittner 2002).

Consider again the judgment [B]: 'Hurricane Walter extends over parts of Wyoming, Montana, Utah, and Idaho' and the corresponding structure $J^A = ([B], Pt^V, Pt^R)$. The *skeleton* of the reference partition Pt^R is the partition Pt^S, which recognizes the United States (Figure 4) and thereby establishes the frame of reference for the approximation. Consider Figure 3. Here the skeleton Pt^S of the reference partitions is an egg-yolk structure containing the cells labeled 'core', 'exterior', and 'where the candidate boundaries are'.

Every reference partition $Pt^R = ((A,\subseteq), (\Delta, \leq), P^R, L^R, \Omega)$ has a crisp partition $Pt^S = ((A, \subseteq), (\Delta, \leq), P^S, L^S)$ called the skeleton of Pt^R. Both, Pt^S and Pt^R share the cell structure $(A, \subseteq)$ and the target domain $(\Delta, \leq)$. In order to ensure that the intuitions sketched in the previous paragraph (and implicitly assumed in our definitions of rough location L^R and rough projection P^R) are satisfied we demand that the skeleton has following properties:

i. If $P^S(z, o)$ holds in Pt^S then so does $P^R(z, o, fo)$ in Pt^R. For all other cells z_1 in the shared cell structure A we have $P^R(z_1, o, no)$. That is:

$$P^S(z, o) \rightarrow (P^R(z, o, fo) \text{ and } (\forall z_1 \in A: z_1 \neq z \rightarrow P^R(z_1, o, no))).$$

ii. If $L^S(o, z)$ holds in Pt^S then so does $L^R(o, z, fo)$ in Pt^R. For all other cells z_1 in A we have $L^R(o, z_1, no)$. That is:

$$L^S(o, z) \rightarrow (L^R(o, z, fo) \text{ and } (\forall z_1 \in A: z_1 \neq z \rightarrow L^R(o, z_1, no))).$$

iii. Skeletons satisfy MB1–6.

Often skeletons are also full, exhaustive, and complete in the sense of (Bittner and Smith 2003), which means in effect that they create subdivisions of the targeted domain into jointly exhaustive and pairwise disjoint portions.

The skeletons of reference partitions which serve as frames of reference are often spatial or temporal in nature. They are relatively stable, i.e., they do not change over time. This implies in turn: (a) that the pertinent cell structure is fixed and (b) that the objects onto which the skeleton projects do not change (they are, for example, spatial regions tied to the surface of the Earth). Consider again the examples in Figure 4. The granular partition projecting onto the United States has existed for more than one hundred years without significant changes. (Hurricane Walter, in contrast, changes continuously throughout the course of its (brief) existence.) In fact, Figure 4 itself needs to be considered as a snapshot of reality at some determinate point in time (Smith and Brogaard, 2002, Bittner and Smith 2003a, Grenon 2003). It provides us with useful information when we are told that Hurricane Walter was located in parts of Montana, Idaho, Wyoming, and Utah at such and such a time. Every American child learns the corresponding reference partition in school, and uses it for all sorts of purposes thereafter (Stevens and Coupe 1978). Reference partitions are characteristically built out of boundaries with which human beings can become easily familiar, objects which facilitate easy learning.

Conclusions

We have proposed an application of the theory of granular partitions to the phenomenon of vagueness, a phenomenon which is itself seen in *de dicto* terms, i.e. as a semantic property of names and predicates. We defended a supervaluationistic theory of the underlying semantics and expressed it in terms of the theory of granular partitions. We showed that the use of frames of reference in making approximating judgments can be formulated very naturally in partition-theoretic terms, and that the framework of granular partitions then helps us to understand the relationships between vagueness and approximation. While the bulk of our examples were derived from the spatial domain, the generality of the theory of granular partitions allows an easy generalization to other sorts of cases.

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References

Bittner, T. (1997). A qualitative coordinate language of location of figures within the ground. In S. Hirtle & A. U. Frank (Eds.), *Spatial information theory: A theoretical basis for GIS*, COSIT 99, 1329 Lecture Notes in Computer Science, Berlin: Springer, 223–240.

Bittner, T. (2002). "Approximate Temporal Reasoning." *Annals of Mathematics and Artificial Intelligenc,. 35*, 1-2

Bittner, T. and B. Smith (2001a). A taxonomy of granular partitions. In D. Montello (Ed.), *Spatial information theory: Foundations of geographic information science*, COSIT 01, Lecture Notes in Computer Science, 2205, Berlin: Springer, 28–43.

Bittner, T. and B. Smith (2001b). Vagueness and granular partitions. In C. Welty & B. Smith (Eds.), *Formal ontology and information systems*, New York: ACM Press, 309–321.

Bittner, T. and B. Smith (2003). A theory of granular partitions. In M. Duckham, M. F. Goodchild & M. F. Worboys (Eds.), *Foundations of geographic information science,* London: Taylor & Francis, 117–151.

Bittner, T. and B. Smith (2003a). Granular spatio-temporal ontologies. *AAAI Spring Symposium on Foundations and Applications of Spatio-Temporal Reasoning* (FASTR).

Bittner, T. and J. Stell (2002). Approximate qualitative spatial reasoning. *Spatial Cognition and Computation*, *2*, 435–466.

Clementini, E. and P. D. Felice (1996). An algebraic model for spatial objects with undetermined boundaries. In P. Burrough and A. U. Frank (Eds.), *Geographic objects with indeterminate boundaries*. London: Taylor and Francis.

Cohn, A. G. and N. M. Gotts (1996). The egg-yolk representation of regions with indeterminate boundaries. In P. Burrough & A. U. Frank (Eds.), *Geographic objects with indeterminate boundaries*. London: Taylor and Francis.

Fine, K. (1975). Vagueness, truth and logic. *Synthese, 30,* 265–300.

Grenon, P. (2003) The spatio-temporal ontology of reality and its formalization. *AAAI Spring Symposium on Foundations and Applications of Spatio-Temporal Reasoning* (FASTR).

Johansson, I. (1989). *Ontological investigations: An inquiry into the categories of nature, man, and society*. New York: Routledge.

Pawlak, Z. (1982). Rough sets. *International Journal of Computation Information. 11*, 341–356.

Roy, A. J. and J. G. Stell (2001). Spatial relations between indeterminate regions. *Journal of Approximate Reasoning,* 27(3), 205-234.

Smith, B. (1995). On drawing lines on a map. In A. U. Frank & W. Kuhn (Eds.), *Spatial information theory: A theoretical basis for GIS,* COSIT 95, Lecture Notes in Computer Science 988 , Berlin: Springer, 475–484.

Smith, B. (2001). Fiat objects. *Topoi, 20(2)*, 131–148.

Smith, B. and B. Brogaard (2002). Quantum mereotopology. *Annals of Mathematics and Artificial Intelligence, 35*, 1–2, 153–175.

Smith, B. and B. Brogaard (to appear). A unified theory of truth and reference. *Logique et Analyse*.

Stevens, A. and P. Coupe (1978). Distortions in judged spatial relations. *Cognitive Psychology, 10,* 422–437.

Varzi, A. (2001). Vagueness in Geography. *Philosophy and Geography, 4*, 49–65.