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Classical Mereology and Restricted Domains

Carola Eschenbach* Wolfgang Heydrich‡

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1 Introduction

Although the notions of *part* and *whole* seem to be complementary, as is especially suggested by the notion of *part-whole relations*, on closer inspection it becomes obvious that this does not quite hold. One simple, but nevertheless important point is that *part* is a (binary) relational concept while *whole* is a (unary) predicative one. Something is a part if and only if it is a part of something; *being a whole* does not mean *being a whole of something(s)*—having parts is not sufficient for being a whole.

To be a whole typically requires having parts with certain properties and organized in such a way that the whole acquires the feature of integrity. Integrity, however, although much discussed in the philosophical tradition since at least Aristotle (cf. Burkhardt and Dufour 1991), is a notion that is not very well understood. It seems to depend on kind, respect, and relevance. Some things may have integrity in one respect but not in another. Several criteria may contribute to the integrity of things—among others: size, connectedness, form and function. Certain parts of some kinds of wholes have to be themselves wholes of a certain kind. Some kinds of wholes require parts which are connected to each other or in certain positions with respect one another. Some parts of a certain whole might be required to be similar in some way or another or to contribute to its function.

The part-of relation, in contrast, is independent of any such criteria of integrity. Being a formal relation, and not being restricted to things of special kinds or specific organization, it is neutral with respect to the choice of a specific domain. The part-of relation is applicable everywhere. Its domain is the completely unrestricted domain of everything. The same is true of its conceptual mates: *proper-part-of* (part of and not identical with), *overlapping* (having a common part), *being discrete* (having no part in common), and *sum* (of one or more things, understood as being composed out of the thing(s) without remainder or addition).

*Universität Hamburg, FB Informatik, Vogt-Kölln-Str. 30, D-22527 Hamburg, Germany. email: eschenbach@informatik.uni-hamburg.de

‡Universität Hamburg, Germanisches Seminar, Von-Melle-Park 6, D-20146 Hamburg, Germany.

These interrelated notions are by now well understood. Their theory is known as classical mereology (CM) which has been given several axiomatizations since its first formulation by Leśniewski (1916). The system we employ, consisting of definitions [D1] – [D6] and axioms [A1] – [A3], corresponds to the one given by Leonard and Goodman (1940).[1] Its primitive notions are that of *discreteness* and *identity*.

[D1] x is *part of* y iff x is discrete from everything y is discrete from.
[D2] x is a *proper part of* y iff x is part of y and y is not part of x.
[D3] x and y *overlap* iff they have a common part.
[D4] x is the *sum* of some entities iff x is discrete from exactly those entities which are discrete from each of them.
[D5] x is the *product* of some entities iff x is the sum of all their common parts.
[D6] x is an *atom* iff it has no proper part.

[A1] x is discrete from y iff x and y do not overlap.
[A2] If x is part of y and y is part of x, then x and y are identical.
[A3] For any entities, their sum exists.

According to [A1] and [D3], *overlapping* and *part-of* can also be taken as primitive notions to describe this structure. Given that

x is part of y iff x is a proper part of y or x and y are identical,
x is part of y iff y is the sum of x and y,
x and y overlap iff the product of x and y exists,

also *proper part*, *sum*, or *product* could be taken as primitive concepts.

In addition to the specific relationships between the concepts involved, CM can be characterized by its extensionality [A2] and completeness [A3]. These principles may become invalid, however, if the quantifiers of CM are restricted in certain ways. But in letting the quantifiers range over everything, as we claim that CM does, it is more plausible to assume that they are well-founded. Both of these principles are based on the formal character of the mereological relations which should be seen in analogy to the relation of identity. Effectively, overlap may be conceived of as partial identity, and being-completely-identical-with may be defined as being part of and having as part. Furthermore, composition (*sum-of*) is a generalization of the ordinary concept of identity. According to this view, there is a unique thing which is the sum of, say, Julius Caesar and Marilyn Monroe. Being their composition without remainder and addition, it is nothing over and above the emperor and the actress. It is simply as one thing what they are as two separate things. Whereas identity is restricted to links between singular terms exclusively, sum-formation provides a link between singular and plural terms as well.

The principle of extensionality [A2] can be paraphrased by: 'if x and y overlap the same

[1] The most important difference between the system of Leonard and Goodman and our formulation consists in our use of plural quantification in e.g. [D4] and [A3] (cf. Lewis 1991, Boolos 1984). We shall make use of this device at several places in this paper. (Cf. our short remarks in paragraph 4 and 5.)

entities (are discrete from the same entities / have the same entities as part / are part of the same entities), then x and y are identical', 'for any objects, their sum is unique', 'no difference without mereological difference', or 'no difference without difference in content'. The principle of completeness or unrestricted sum-formation [A3] specifically corresponds to the view of composition as identity that is argued for by Armstrong (1978), Baxter (1988a,b) and Lewis (1991).

CM as the general theory of formal mereological concepts does not have any specific notion of *integrity* or *being a whole*. This neutrality should not be seen as a disadvantage of the theory but as displaying its power: the concepts of CM are applicable in each and every domain, hence its proper domain is the unrestricted universe of everything and, accordingly, the quantifiers of CM are kept extremely open.

This point of view is not generally accepted. Researchers in several fields have found it appropriate to introduce domain-specific, quasi-mereological concepts of *being-part-of*, *overlapping*, *being-discrete* or the like. Typically, these quasi-mereological concepts are definitionally connected with certain explicitly non-mereological terms. It seems that in such approaches the question of whether something fulfills a certain criterion of integrity has come to be mixed up with the question of whether it exists. Although we have been rather skeptical with respect to applying CM to certain domains ourselves (Heydrich 1988, Eschenbach et al. 1990), we now think that taking CM for granted in looking at different domains can shed more light on the specific nature of these domains, their similarities and differences. We claim that the differences are not based on different mereological concepts but on different concepts of integrity or being a whole.

In the following sections, we briefly describe axiomatic accounts of three special domains. None of these restricted domains can be considered as a model of CM, but each of the accounts makes use of specific quasi-mereological concepts. *Quasi-mereological* because these concepts show enough similarity to the family of classical concepts by the way they are interrelated with one another. However, either the principle of extensionality, the principle of unrestricted sum-formation or one of the classical interrelations between the mereological concepts has to be abandoned. Nevertheless, we show that CM is applicable to these domains as soon as they are seen as being embedded in a less restricted (or even the most comprehensive) domain.

The first account concerns linear orders of extended entities as they can be found in discussions of the ontology of time. The non-mereological domain-specific concept in this case is the relation of total precedence. In restricting the account to things which bear integrity in the sense of connectedness, one has to abandon the principle of unrestricted sum-formation. Considering the comprehensive domain, however, allows one to keep CM intact and to describe the structure of the domain by connecting the notion of precedence with the general notions of CM.

The second account deals with topological structure. Our starting point is Clarke's approach of defining quasi-mereological notions on the basis of *connection*, which is Clarke's topological primitive (Clarke 1981). The quasi-mereological structure thereby defined diverges from CM in that the role of overlap is taken over by two notions. Only one of them satisfies the condition that x is part of y if and only if everything overlapping x overlaps y, and only the other one satisfies the condition that overlapping implies having a common part. We show that it is not necessary to abandon CM in order to define Clarke's topological structures. On the contrary, one may arrive at a considerable conceptual simplification by embedding both domains in a less restricted (or even the most comprehensive) domain. We choose *region* as our primitive and show how to regain Clarke's structure by connecting this notion with the general conceptual apparatus of CM.

The third account takes up Lewis' recent treatment of set-theory (ST). In ST we find the quasi-mereological notions of *subclass* (part), having a *non-empty intersection* (overlap), *union* (sum), etc. Given this interpretation, the structure defined by the axioms of ST does not satisfy CM for two reasons. First, having a common part does not imply overlap. Second, the existence of arbitrary sums of sets is not guaranteed in the domain of sets. Following Lewis (1991), we show how to disentangle specific set-theoretical concepts from the general notions of CM. This leads to the singleton function as a new primitive. The domain of sets is embedded in the comprehensive universe of everything: individuals, classes, and their sums. In our view, the concept of integrity inherent in the notion of sets, as reconstructed by Lewis, turns out to be a matter of *size*.

2 Linear orders of extended entities

Orders of extended entities as defined by [D10] can be important in the study of ontologies such as space, time, and situations. Since these orders have not been much investigated, several definitions of properties of relations have to be generalized to be applicable in a reasonable way both to them and to orders of non-extended entities. For example, the definition of *external* in [D8], which is due to Leonard and Goodman (1940), generalizes the definition of irreflexivity, corresponding to the view of *overlapping* as partial identity. To distinguish e.g. temporal structure from causal structure, it is necessary to establish a notion of linearity for such orders as well as for orders of non-extended entities. Taking [D11] as the definition of *exhaustion*, linearity of orders of non-extended entities can be defined by [D12]. With respect to external orders of extended entities one has to generalize the notion of linearity since such an order R cannot relate an R-extended entity and a proper part of it.

[D7] A (binary) relation is a *(partial) order* iff it is transitive and irreflexive.

[D8] A relation is *external* iff no overlapping entities are related by it.

[D9] Let R be an order. An entity is *R-extended* iff it has parts x and y such that xRy.

[D10] An order R is an *order of extended entities* iff R has some R-extended entity in its field. Otherwise R is an *order of non-extended entities*.

[D11] A relation R *exhausts* the (symmetric) relation S iff xSy implies xRy or yRx.

[D12] An order of non-extended entities is *linear* iff it exhausts non-identity.

Orders of extended entities are well known in the discussion of temporal ontology. Although at the first glance, the structure of time seems to be quite simple (e.g. compared with the structure of space or situations) a lot of different models have been proposed for it. The differences are mainly due to assumptions whether time spans and/or time points exist, whether time spans have to be bounded, whether the flow of time is linear or might be branching, or whether the differences of temporal entities can be described by means of their temporal relations to other temporal entities (extensionality). The difference between time spans and time points should, in our view, be drawn by using mereological notions: while the former may have proper parts which are temporal entities, the latter do not. In this section, we will take the domain of temporal entities as an example of a linear, external order of extended entities, since the underlying ontological structure belongs to the most prominent ones and seems to allow for the definition of mereological relations (cf. van Benthem (1983), Allen (1984)).

The structure of the domain of temporal entities can be described on the basis of the binary relation *total temporal precedence*. There are, of course, other notions of temporal precedence which might e.g. allow for the overlap of ordered entities. But any presentation of temporal structures should make the notion of total temporal precedence available, i.e. should take it as primitive or allow for its definition. Since our main concern is that of ontological structure, our discussion applies not only to those approaches taking total precedence as primitive but to all approaches which allow for its definition.

Total precedence being an external order, we will assume [A_{PR}1] and [A_{PR}2] to hold.

[A_{PR}1] If x precedes y and y precedes z, then x precedes z.
[A_{PR}2] Every entity is discrete from any entity it precedes.

According to the view of *overlapping* as partial identity, it seems straightforward to employ an axiom like 'If x and y are discrete, then x precedes y or y precedes x' (i.e. precedence exhausts discreteness) as a generalization of the definition of linearity (cf. Benthem 1983). In the context of [A_{PR}2], this allows for the definition of such domain-dependent quasi-mereological notions as *p-part*, *p-overlapping*, *p-discreteness* and *p-sum* ([D_{PR}2] – [D_{PR}4]) on the basis of *precedence*, employing well known interrelations of CM. Axiom [A_{PR}2'] can now replace [A_{PR}2], so that the system becomes independent from CM.

[D_{PR}1] x is *p-part* of y iff every entity which precedes y or is preceded by y precedes x or is preceded by x.
[D_{PR}2] x *p-overlaps* y iff x and y have a common p-part.
[D_{PR}3] x is *p-discrete* from y iff x and y do not p-overlap.
[D_{PR}4] A *p-sum* of some entities is p-discrete from z iff all of them are p-discrete from z.

[A_{PR}2'] x and y are p-discrete iff x precedes y or y precedes x.

The similarity between CM and the mereological structure defined by [D_{PR}1] – [D_{PR}4], [A_{PR}1] and [A_{PR}2'] is not superficial. E.g. *p-part* is reflexive and transitive, *p-overlapping* is symmetric and reflexive, *p-discreteness* is symmetric and irreflexive, and *being a p-part* of some entity x means not p-overlapping any entity x is p-discrete from. Furthermore, adding a precedence-based principle of extensionality such as 'If x and y precede the same entities and are preceded by the same entities then they are identical' will result in the structure being extensional with respect to the quasi-mereological notions, e.g. *p-part* becomes anti-symmetric and *p-sum* becomes unique. The only difference from CM is that in the structures as defined here, the p-sum of two (or more) entities need not exist. In addition, the assumption that arbitrary p-sums exist is incompatible with the assumption that more than three entities exist.

This is a consequence of the underlying assumption that all entities have to be convex with respect to precedence, which is built into [A_{PR}2'] (as well as the formulation of the principle of extensionality just mentioned). Loosely speaking, an entity is convex if it has no holes, i.e. it has everything which is (with respect to precedence) between two p-parts of it as p-part. It is provable that the structure as defined above does include only such entities.

Convexity seems to be a criterion of integrity. We do not expect every sum of convex entities, i.e. their composition without remainder and addition, to be convex. Thus, the difference between *part-of* and *p-part* seems to be that the latter is a kind of part-whole relation. But the domain as described here does not allow for distinguishing these two kinds of relations, since every entity is assumed to be a whole (i.e. to be convex). Consequently, there are no non-convex sums of convex entities: Not to bear integrity means not to exist. As a consequence, *p-part* cannot be conceived of as a restriction of *part-of* to a specific domain (of wholes).

Fortunately, we can easily get rid of the domain-dependent quasi-mereological notions if we allow non-convex entities to exist. It is not our point to propose a temporal ontology of non-convex entities as an alternative to ontologies of points and convex time-spans. Our question is whether it is in general necessary to deny the existence of non-convex extended entities for the sake of assuming a domain-specific external order to be linear.

We start out by assuming the mereological relations to be defined by CM, independently from any domain-specific notion. A general definition of *R-betweenness* and *R-convexity* on the basis of *part-of* and order R are then provided by [D13], [D14], and the notion of being *R-disentangled* is added by definition [D15]. Linearity of orders of extended entities can now be defined in a manner not assuming convexity of everything [D16], which like [D12] is a special case of [D17].

Let R be an order.

[D13] x is *R-between* y and z iff yRx and xRz, or zRx and xRy holds.

[D14] x is *R-convex* iff every entity which is R-between two parts of x is part of x.

[D15] x and y are *R-disentangled* iff they are discrete and neither of them has a part which is R-between two parts of the other.

[D16] An external order of extended entities R is *linear* iff it exhausts R-disentangledness.

[D17] An external order R is *linear* iff it exhausts R-disentangledness.

In the domain of temporal entities, [D17] becomes [$A_{PR}3$][2], while [$A_{PR}1$] and [$A_{PR}2$] do not need revision. With respect to precedence, the substructure of p-convex entities is isomorphic to the structure defined earlier, but it allows for the existence of non-p-convex entities and thus for the existence of arbitrary sums of temporal entities.

[$A_{PR}1$] If x precedes y and y precedes z, then x precedes z.
[$A_{PR}2$] Every entity is discrete from any entity it precedes.
[$A_{PR}3$] If x and y are p-disentangled, then x precedes y or y precedes x.

Although the structure allows for the existence of non-p-convex entities, it is still possible to reject the assumption that non-p-convex temporal entities exist. Defining the notion of *temporal entity* by [$D_{PR}5$] and replacing [$A_{PR}3$] by [$A_{PR}3$'] yields a basis for restricting the domain of the relation of precedence without giving up a clear notion of linearity. Thus, taking CM as the general theory of mereological concepts and at the same time accepting the existence of non-p-convex entities does not mean assuming the existence of non-p-convex temporal entities. Instead, we can find a way of defining criteria of integrity (as p-convexity) according to assumptions concerning the restrictions on entities belonging to certain domains.

[$D_{PR}5$] x is a *temporal entity* iff it is in the field of the precedence relation.
[$A_{PR}3$'] If x and y are p-disentangled temporal entities, then x precedes y or y precedes x.

As stated above, it is not our purpose to propose a specific ontological structure as being the 'true' structure of time. On the contrary: the structure discussed here is neutral with respect to a lot of interesting questions concerning the natural ontology of time: In which cases is the sum of temporal entities a temporal entity? Are there temporal entities which are atoms? Are there time-points (i.e. temporal entities which have no temporal entities as proper parts)? Is there a beginning (end) of time (i.e. a temporal entity which precedes (is preceded by) every other temporal entity)? Is the past (future) of a temporal entity a temporal entity (i.e. the sum of all entities preceding (being preceded by) it)?

Instead our goal was to disentangle the relation-theoretic notion of linearity from ontological assumptions in order to get rid of domain-specific quasi-mereological notions, and thus to reach a level of neutrality with respect to assumptions concerning features of integrity in the temporal domain. The addition of specific restrictions on temporal entities to the system obtained will yield more specific domains, which may be more 'natural' with respect to how we conceive of time.

[2]Using *p-disentangledness* as an abbreviation for *precedence-disentangledness* etc.

3 Quasi-Topology

While time is usually assumed to be one-dimensional, an assumption that is reflected by assuming temporal *precedence* to be a linear order, the structure of space is more complex. Topology, usually formulated on the basis of set theory, reflects some important aspects of this structure and is neutral with respect to the number of dimensions. Clarke (1981) presented an axiomatic system which is quite close to topology. His system is based on the primitive binary relation of *connection*, which is used for the definition of quasi-mereological notions. In Clarke (1985), he modified the system slightly and extended it by presenting a definition of *point* on the basis of set-theoretic constructions and by adding an axiom to secure the existence of points with certain properties. To distinguish between quantifying over regions (in his terms: individuals), points and sets of points, Clarke uses different kinds of variables according to the difference in type. Since points are not considered to be individuals, they are not embedded into the mereological structure but are related to regions by a non-mereological relation *in*.

In this section we will concentrate on the topological structure defined by Clarke in 1981, although it would be interesting to discuss how his notion of *point* can be regained on the same level as that of *region*, such that points are parts of the regions they are in. But in studying his approach we found the unfortunate result that—contrary to what Clarke intents—the calculus he defined in 1985 reduces to the classical calculus of mereology.[3]

Topology gives rise to several concepts of integrity (*self-connectedness, lack of holes*, etc.). They are definable in terms of more basic concepts, which we will concentrate on in the following discussion. Our main point is how the system developed by Clarke (1981) on the basis of topological notions can be embedded into a classical mereological framework.

The presentation of the structure in Clarke (1981) is divided into three subsections, called 'Mereological part', 'Quasi-Boolean part', and 'Quasi-Topological part', respectively. The 'Quasi' in the latter names is motivated by the non-existence of an *empty individual* (corresponding to the empty set in set-based approaches), which leads to the partiality of functions such as *product* or *complement*. However, the difference between CM and Clarke's connection-based mereological system is more striking than the difference between classical topology and his quasi-topology. To give the reader an idea of this, we will present some of Clarke's definitions (with a slightly changed terminology).

[$D_{QT}1$] x is an *r-part* of y iff y is connected to every region x is connected to.
[$D_{QT}2$] x is a *proper r-part* of y iff x is an r-part of y and y is not an r-part of x.
[$D_{QT}3$] x *r-overlaps* y iff x and y have a common r-part.
[$D_{QT}4$] x is the *r-sum* of some regions iff x is connected to exactly those regions which are connected to at least one of them.
[$D_{QT}5$] The *r-product* of some regions is the r-sum of the common r-parts of them.
[$D_{QT}6$] x is *r-discrete* from y iff x and y do not r-overlap.

[3]This observation will be the topic of a future paper. The proof is too complex to present here, especially since this paper is not devoted to specifically topological questions.

[$D_{QT}7$] x and y are *externally connected* iff they are connected and r-discrete.
[$D_{QT}8$] x is a *tangential part* of y iff x is an r-part of y and externally connected to a region which is externally connected to y.
[$D_{QT}9$] x is a *non-tangential part* of y iff x is an r-part of y and not externally connected to any region which is externally connected to y.
[$D_{QT}10$] The *interior* of x is the r-sum of the non-tangential parts of x.
[$D_{QT}11$] x is *open* iff it is identical with its interior.

The axiomatic system given in Clarke (1981) corresponds to the system [$A_{QT}1$] – [$A_{QT}6$]. Our explicit restriction of the quantification to regions captures Clarke's use of just one sort of variables.

[$A_{QT}1$] Every region is connected to itself.
[$A_{QT}2$] If x is connected to y then y is connected to x.
[$A_{QT}3$] If x and y are connected to the same regions, then they are identical.
[$A_{QT}4$] For any regions their r-sum exists.
[$A_{QT}5$] Any region has a non-tangential part.
[$A_{QT}6$] If x and y are not externally connected to any region and r-overlap, then their r-product is not externally connected to any region.

Comparing these definitions and axioms with the mereological ones, we see that *connection* takes over the role of *overlap* in CM with respect to the definitions of *r-part*, *r-sum* and extensionality [$A_{QT}3$]. But in contrast to *r-overlap*, *connection* does not guarantee the existence of a common r-part. With respect to the quasi-mereological relations of *r-overlap* and *r-discreteness*, the structures defined are not extensional. But the identification of *connection* and *r-overlap* would result in the emptiness of the notion of *external connectedness* and *tangential part*. As a consequence, every region would be open and all the topological distinctions would vanish (i.e. be reduced to mereological ones).

As already obvious in the title ('A Calculus of Individuals Based on "Connection"'), Clarke assumes the topological notions to be more basic than the mereological ones. As a consequence of his definitions it becomes necessary to assume that points and boundaries are not embedded in the mereological structure but are a kind of second class citizens of his theory, or, as Simons (1987: 98) puts it: 'What we are being asked to believe is that there are two kinds of individuals, "soft" (open) ones, which touch nothing, and partly or wholly "hard" ones, which touch something. Yet we are not allowed to believe that there are any individuals which make up the difference.'

Assuming mereology to be more basic than topology, we want to show that the possibility of introducing topological notions is not only given by taking *connection* as the basis. In contrast to the approach of Smith (1993), however, we will aim thereby at specifying the same structure as Clarke. A topological concept which is even more basic than *connection* is *region*. That this is true for Clarke's framework as well is obvious in his restricting quantification by using different kinds of variables in the later paper. We claim that the distinction between regions and non-regions already allows for the establishment of the whole framework of quasi-topology on the basis of CM. Thus, the differences between different kinds of topological structures can be described as differences regarding the underlying concept of region. Nevertheless, some restrictions must be fulfilled by any such concept. We start out by giving new definitions of *connection*, *r-overlapping* (overlapping by a region), *external connectedness*, and *openness*, assuming CM as the framework.

[D_{QT}0] x and y are *connected* iff they are overlapping regions.
[D_{QT}3'] Regions x and y *r-overlap* iff they have a common part which is a region.
[D_{QT}4'] Region x is the *r-sum* of some regions iff x is connected to exactly those regions which are connected to at least one of them.
[D_{QT}5'] The *r-product* of some regions is the r-sum of their common parts which are regions.
[D_{QT}6'] Regions x and y are *r-discrete* iff they do not r-overlap.
[D_{QT}7'] Regions x and y are *externally connected* iff they are connected and r-discrete.
[D_{QT}8'] Region x is a *tangential part* of region y iff x is part of y and externally connected to some region which is externally connected to y.
[D_{QT}9'] Region x is a *non-tangential part* of region y iff x is part of y and not externally connected to any region which is externally connected to y.
[D_{QT}10'] The *interior* of region x is the r-sum of the non-tangential parts of x.
[D_{QT}11'] A region x is *open* iff it is identical with its interior.

The topological structure of the system presented in Clarke (1981) can be regained by means of [A_{QT}3'] – [A_{QT}6']:

[A_{QT}3'] If every region that overlaps region x overlaps region y, then x is part of y.
[A_{QT}4'] For any regions their r-sum exists.
[A_{QT}5'] Every region has an open region as a part.
[A_{QT}6'] The r-product of any two overlapping open regions is an open region.

While [A_{QT}4'] and [A_{QT}6'] seem to be essential for calling the system *quasi-topological*, the other axioms could be modified or abandoned in order to derive different systems.

Although not every region needs to be open, open regions in the topological structure show few signs of integrity. Arbitrary sums of open regions are open regions and open regions overlap if and only if they have an open region as a common part. In the structure defined here, the remainder principle also holds: if an open region x is not part of an open region y, then there is an open region which is part of x and discrete from y. Thus, the structure of open regions is a model of CM. While arbitrary sums of regions are regions, and mereological extensionality holds according to [A_{QT}3'] for the structure of regions, overlapping regions need not have a region as a common part and the remainder principle need not hold (i.e. a region which is not part of a region x need not have a region discrete from x as part). Thus the concept of region exhibits more features of integrity than the concept of open regions does.

4 Set Theory

Traditionally, set theory (ST) and classical mereology (CM) are conceived of as basically incompatible accounts. Leśniewski developed CM as a formal alternative to axiomatic ST in order to arrive at a paradox-free notion of classes within his general system, and Goodman (1964) has argued that fundamentally different generating relations are constitutive for the domains of ST and CM.

In his recent book 'Parts of Classes', David Lewis has challenged this assumption. Taking seriously the view that CM is concerned with the completely unrestricted domain of everything, he argues that sets can be conceived of as a proper sub-domain within the all-embracing universe of things. According to him, the conceptual apparatus of CM is applicable to sets as well: sets (like everything else without restriction) have parts and are parts. They overlap or are discrete, and they have sums. Whereas, however, we have the general principle that sums of things are things, the principle that sums of sets are sets is not valid. But this does not mean that we have to give up unrestrictedness of sum-formation as a general principle when dealing with the domain of sets. Sums of sets exist (they are things). Sometimes they are sets themselves (thus within the domain of specific interest in set theory); sometimes, however, they are proper classes (thus outside the interesting domain). Insofar as sets cannot be arbitrarily composed (summed up) to form further sets, there seems to be a special notion of integrity inherent in the notion of sets.

As it turns out, integrity of sets seems to be mainly a matter of *size*. There is a popular parlance in expositions of ST, according to which some classes are 'too large' to be sets. Following Lewis, one may take this metaphor quite literally. He distinguishes between small things and large things basically on mereological grounds. Given this distinction, sets are small and proper classes are large.

The fundamental non-mereological relation which gives rise to this kind of integrity for things in the domain of sets is, according to Lewis, the relation between a unit class (a singleton) and its member or—given the one-one character of this relation—the (partial) singleton function which assigns unit classes to individuals and sets (but to nothing else).

In order to be more explicit about Lewis' account, let us consider first three (somewhat idiosyncratic) definitions of *class*, *set* and *individual* taking *member-of* and *null-set* as primitives.

[$\mathrm{D_{ST}}1$] x is a *class* iff it has members.
[$\mathrm{D_{ST}}2$] x is an *individual* iff it is a member without members.
[$\mathrm{D_{ST}}3$] x is a *set* iff x is identical with the null set or x is a class that is a member of some class.

The idiosyncrasy simply consists in taking the null set to be a set but not a class. This, however, is merely terminological, since it does not prohibit a standard axiomatic account of ST. (The concepts of *ordered pair*, *non-empty intersection*, *subset*, *union* and *nesting with no greatest member* are understood in the usual way.)

[$\mathrm{A_{ST}}$1] The null set is a set with no members.

[$\mathrm{A_{ST}}$2] No two classes have the same members; no class has the same members as the null set.

[$\mathrm{A_{ST}}$3] If each of x and y is an individual or a set, then there exists a set of x and y.

[$\mathrm{A_{ST}}$4] Given a set x, and given some things, there is a set of all and only those of the given things that are members of x.

[$\mathrm{A_{ST}}$5] If there are some ordered pairs whereby each member of a class x is paired with exactly one member of a class y, and if for each member of y there is a member of x that is paired with it, and if x is a set, then y is a set.

[$\mathrm{A_{ST}}$6] No class has a non-empty intersection with each of its members.

[$\mathrm{A_{ST}}$7] Suppose x is a class, and suppose there are some ordered pairs whereby each member of x is paired with at least one thing, and no two members of x are paired with the same thing. Then there is a class y such that each member of x is paired with exactly one member or y.

[$\mathrm{A_{ST}}$8] If x is a set, there is a set of all subsets of x.

[$\mathrm{A_{ST}}$9] If x is a set, there is a set of all members of members of x.

[$\mathrm{A_{ST}}$10] There is a nesting with no greatest member, the union of which is a set.

Let us note, now, that it is possible to redefine all the well-known set theoretical notions directly in terms of the singleton function and mereological notions. Here are some prominent examples:

[$\mathrm{D_{ST}}$1'] A *class* is a sum of singletons.

[$\mathrm{D_{ST}}$0.1] x is a *member* of y iff y is a class and the singleton of x is part of y.

[$\mathrm{D_{ST}}$2'] An *individual* is anything that has no singleton as part.

[$\mathrm{D_{ST}}$0.2] The *null set* is the sum of all individuals.

[$\mathrm{D_{ST}}$3'] x is a *set* iff it is identical to the null set or a class that has a singleton.

[$\mathrm{D_{ST}}$4] x *includes* y iff (1) y is the null set and x is the null set or a class, or (2) y and x are classes and y is part of x.

[$\mathrm{D_{ST}}$5] x is a *subclass* of y iff x is a class and y includes x.

[$\mathrm{D_{ST}}$6] x is an *urelement* iff it is an individual and not a set.

[$\mathrm{D_{ST}}$7] A *proper class* is a class without a singleton.

Note that [$\mathrm{D_{ST}}$1] and [$\mathrm{D_{ST}}$1'] define the same idiosyncratic notion of class. The null set is—according to [$\mathrm{D_{ST}}$0.2]—the most comprehensive member without members. It is both an individual and a set, but not a class or urelement. Although included in every class, it is not a subclass of any.

Lewis' main thesis is the following almost trivial claim:

[$\mathrm{T_{ST}}$1] The parts of a class are all and only its subclasses.

Since the null set is not a class, it follows that singletons, having no parts except themselves, are atoms, and, effectively, that sums of individuals are individuals. Furthermore, only individuals are parts of individuals, such that individuals are different only if they overlap different individuals.

Consequently the general principles of unrestricted sum-formation and extensionality are valid in the domain of individuals. In this respect the restricted domain of individuals is like the unrestricted universe. This indicates that there are no special conditions of integrity for being an individual (hence, presumably, the somewhat misleading label 'calculus of individuals' for CM). As for classes (in Lewis' sense), the principles of unrestricted sum-formation and extensionality are valid as well: sums of classes are classes, and classes are different only if they overlap different classes.

Given the unrestricted domain of CM there must still exist things beyond the mutually disjoint domains of individuals, non-empty sets and proper classes. There are sums of classes and individuals: mixed things with singletons and individuals as parts. These mixed sums are, according to Lewis (and in contradistinction, say, to Bunt's ensemble theory (1985)), excluded from set membership, hence from the domain of the singleton function. This feature, however, does not seem very essential for the account. The singleton function has to be partial anyway, since otherwise, Russell's paradox threatens. Some sums of singletons have to be proper classes; they are not a member of anything, and hence have no singletons.

It is common in set theory to differentiate between members (sets and, if any, urelements) and non-members (proper classes) by means of axioms like [$A_{ST}1$] – [$A_{ST}10$] in place of the paradoxical principle of unrestricted set comprehension. As Lewis' analysis (see especially [$D_{ST}0.2$] and [$T_{ST}1$]) shows, the formulation of these axioms intertwines mereological notions with assumptions about singleton formation. Lewis separates these components thoroughly by giving a self-sustained axiomatic account of the singleton function, which is called 'Mereologized Arithmetic' (MA), since it turns out to be a generalized version of Peano's arithmetic:

[$A_{MA}1$] Nothing has two different singletons.

[$A_{MA}2$] Any part of the null set has a singleton; any singleton has a singleton; any small sum of singletons has a singleton; and nothing else has a singleton.

[$A_{MA}3$] No two things have overlapping singletons, nor does any part of the null set overlap any singleton.

[$A_{MA}4$] If there are some things, if every part of the null set is one of them, if every singleton of one of them is one of them, and if every sum of some of them is one of them, then everything is one of them.

The null set takes the place of zero in MA, and the singleton function takes the place of the successor function. According to [$A_{MA}1$] *singleton-of* is a partial function. Its domain is specified in [$A_{MA}2$]. According to [$A_{MA}3$] singletons are atomic classes (since they are discrete both from each other and the null set and unrestrictedly everything is composed out of parts of the null-set and singletons). Finally, [$A_{MA}4$] parallels the axiom of induction.

Lewis' aim is to regain set theory, i.e. to prove the axioms of ST. But, obviously, MA (the pure theory of the singleton function) plus CM are not strong enough for this. E.g., they do not exclude unintended domains with only countably many things, whereas ST presupposes many more.

What is required is a framework which embeds MA and provides the requisite logical strength. This framework comprises four components: (a) first order logic with identity, (b) CM, (c) a device of plural quantification, and (d) some special assumptions.

(a) and (b) need no comment here. As for (c), however, it needs mention that the device of plural quantification—used already by Leśniewski and rediscovered by Boolos (1984)—confers the logical strength of monadically quantified second order logic to Lewis' framework—without, however, any commitment to classes.

Note that plural quantification has already been used above e.g. in our formulations of the axioms [A3], [A_{ST}4], [A_{ST}5], [A_{ST}7], and [A_{MA}4]. It plays its role as well in the definition of *small* sums of singletons, which turns out to be of crucial importance for MA. (Cf. the reference to 'small sums of singletons' in [A_{MA}2].)

The predicate *small* is defined in tandem with *large*. Likewise, Lewis defines *finite* and *infinite* as well as the plural predicates *few* and *many*.

[D18] x is *large* iff there are some things such that (1) no two of them overlap, (2) their sum is the whole of Reality, and (3) each of them contains exactly one atom that is part of x and at most one other atom.

[D19] x is *small* iff it is not large.

[D20] x is *infinite* iff x is the sum of some things, each of which is a proper part of another.

[D21] x is *finite* iff it is not infinite.

[D22] Suppose we have some things such that some large thing does not overlap any of them. Then they are *few* iff there is some small thing x, and there are some things, such that (1) x does not overlap the sum of the former things, (2) each of the latter things is the sum of one of the former things and one atom of x, (3) for each of the former things, one of the latter things is the sum of it and one atom of x, (4) and no atom of x is part of two or more of the latter things.

[D23] Suppose we have some things such that some large thing does not overlap any of them. Then they are *many* iff they are not few.

Something is *small* iff (rendered informally) it has fewer atoms than there are in all the rest of the universe. Otherwise it is *large*. And some things are *few* iff (rendered informally again) they are less than there are atoms in all the rest of the universe. Otherwise they are *many*. Note that the conceptual means for the definitions [D18] – [D23]—taken from the components (a), (b) and (c) of the framework—do not presuppose the existence of classes. These components alone, however, are not strong enough to derive all the axioms for ST from MA. Some special assumptions are still needed. They are component (d) of Lewis' framework. Lewis formulates five schemata (two for choice, [S1] and [S2], two for replacement, [S3] and [S4], and a so-called Dedekind schema, [S5]) as well as three hypotheses concerning the size of the all-embracing universe. Like plural quantification, these assumptions are not committed to classes. First, the schemata (in which '...' is a place holder for binary relations):

[S1] If there are some things, and each of them ... some things, and no two of them ... the same things, then there are some things such that each of the former things ... exactly one of the latter things.

[S2] If nothing ... itself, and if whenever x ... y and y ... z then x ... z, and if there are some things such that each of them ... another one of them, then also there are some of those things such that (1) among the latter things also, each one ... another one; (2) whenever x and y are two of the latter things, then either x ... y or y ... x.

[S3] If each atom of a thing x ... exactly one atom of a thing y, and if for each atom of y there is an atom of x that ... it, and if x is small, then y is small.

[S4] Given some things, and given some other things (not necessarily different), if each of the former things ... exactly one of the latter things, and if for each of the latter things there is one of the former things that ... it, and if the former things are few, then the latter things are few.

[S5] If x is a proper part of y, and if each atom of y ... exactly one atom of x, and if each atom of x is such that exactly one atom of y ... it, then y is infinite.

And now the hypotheses:

[H1] If something is small, then its parts are few.

[H2] If some things are small and few, their sum is small.

[H3] Some sum of atoms is infinite and yet small.

The point of the three hypotheses is this: Without [H1], [H2] and [H3] the framework of Lewis' theory allows domains with countably many things for MA. Effectively, in countable domains small things are just finite things, and large things are just infinite things with only countably many parts. But given [H1], [H2] and [H3], countable domains are excluded. The universe exceeds each fixed cardinality, being large enough to embed the intended domain of ST. Effectively, the sub-domain of sets in the all-embracing universe of everything is the cumulative hierarchy of ST (with urelements).

Our claim, that integrity of things in the domain of sets is a matter of *size* should be by now straightforward. Only small things have singletons. Singletons, being atoms, are small. And so are sets in general, since each set is either identical with the null-set (which is small because it has at most few atoms as parts) or a small sum of singletons. Sums of some sets which are many exist as well, but they are large. Hence, they have no singleton, and are not members of any class. They are proper classes. The domain of things comprising the null set and all the classes divides into two sub-domains: the sub-domain of large things (i.e. proper classes) and the sub-domain of small things (i.e. sets).

5 Conclusion

We have compared more or less standard accounts of time-ontology, quasi-topology and set-theory under the perspective of mereology. At first glance, it seems that in developing these accounts one can make use of the conceptual apparatus of mereology only in some analogical and distorted sense, and that one is concerned with quite separate and hardly comparable domains. On closer inspection, however, we arrive at the conclusion that the accounts under consideration can be seen as dealing with one and the same comprehensive domain—although under certain characteristic restrictions and focusing on specific aspects. The notions of mereology which are constitutive for the unrestricted universe of everything are applicable in all three areas of formal investigation without modification or distortion.

Each of the accounts may be axiomatically formulated by adding one non-mereological primitive to whatever concepts are chosen to develop CM. In the case of time-ontology we took the two-place predicate *precedence* as a new primitive, in the case of quasi-topology we took the one-place predicate *region* and in the case of set-theory we followed Lewis in using the singleton function. In each case we could show that we can rely on the same formal framework, namely first order logic with identity plus CM. In the case of set-theory, however, it turned out that we need a somewhat extended framework providing additional logical strength.

We get this strength by adding plural quantification to the framework together with some principles mainly concerning interrelations between the notions of mereology and the device of plural quantification. The device of plural quantification in itself is useful already within CM alone, since it allows us to formulate a completely general version of the axiom of unrestricted fusion which is free of any set-theoretical assumptions. One final addition to the framework still concerns three hypotheses as to the size of the universe, guaranteeing it to be large enough to embed every set of the ZF-hierarchy.

Note that the additions to the framework (although necessary for the development of ST), do not presuppose anything specific about sets or, in fact, anything specific about anything else. The extended framework is as topic-neutral, precise, well-understood and ontologically innocent as the unextended framework alone. Whatever there may be within set theory that is philosophically problematic and ontologically obscure, it does not hinge—at least according to Lewis—upon anything from the framework. The obscurity of set theory is the obscurity of the singleton-function exclusively and this function is studied (but arguably not sufficiently understood) in mereologized arithmetic.

Actually, we could have chosen the extended framework already in our examples of time-ontology or quasi-topology. But, its additional logical strength is simply superfluous in these two cases. However, the demonstrated possibility of embedding the divergent accounts in one formal framework strongly recommends CM as the formal basis of what might be called 'Natural Ontology.' This endeavour concerns the formal analysis and reconstruction of schemata of basic categorization. In Natural Ontology we are dealing with such fundamental categories as things and space, actions, events, processes, states and time, matter, stuff and qualities.

The fundamental categories of Natural Ontology exhibit specific material features of integrity. We did not contribute much to their elucidation in this paper. Instead we studied the somewhat less natural, so to speak 'artificial', ontologies of simple, but formally interesting structures (linear orders, topology, set theory). We started by pointing out that, whereas we have (by virtue of CM) a sufficiently clear and precise understanding of the part-of relation, our understanding of the part-whole relation (or of part-whole relations) is still in bad shape. And the same is true of the closely associated notion of integrity. Although its material aspects do not lie within the scope of this investigation, there are, it seems to us, some formal aspects of integrity that arise from our account.

If, for example, we study a non-empty domain of convex entities within the comprehensive universe of CM by using the primitive concept of total precedence, we observe that this domain does not constitute a model for CM. However, the domain of sums of such entities does. This might be taken as reflecting, in a way, that the former entities are conceptually more integrated than the latter. Likewise: If we study a non-empty domain of regions within the comprehensive universe, we find that this domain need not be a model of CM. However, the domain of regions, boundary elements and their sums is such a model. This again reflects an increase of conceptual integration of regions in contradistinction to arbitrary sums and products. Finally, studying sets as a non-empty domain within the comprehensive universe by means of the primitive concept of singleton, we find that they do not constitute a model of CM, whereas e.g. the domains of individuals and classes (or sums thereof) are such models. We take this as indicating that sets are conceptually more integrated than mere individuals, classes or arbitrary entities.

One pertinent observation here is the following: whenever we are able to conceptualize a domain of nice entities within the comprehensive universe (where by 'nice entities' (cf. Lewis 1991: 22) we understand things such that every two of them are mutually discrete—such as atoms or elephants or points) this domain does not constitute a model of CM, whereas, of course, the domain of their sums will always be such a model. This indicates that nice things exhibit a kind of integrity which arbitrary entities lack.

Observations like this, it seems to us, might be taken as a starting point for a formal theory of integrity.

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