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A Spatial Logic based on Regions and Connection

David A. Randell, Zhan Cui and Anthony G. Cohn

Division of Artificial Intelligence
School of Computer Studies
Leeds University
Leeds, LS2 9JT, England
{dr, cui, agc}@scs.leeds.ac.uk

Abstract

We describe an interval logic for reasoning about space. The logic simplifies an earlier theory developed by Randell and Cohn, and that of Clarke upon which the former was based. The theory supports a simpler ontology, has fewer defined functions and relations, yet does not suffer in terms of its useful expressiveness. An axiomatisation of the new theory and a comparison with the two original theories is given.

1 Introduction

The use of interval logics for the representation of time are well known in AI research - see for example Allen (1984) and Allen and Hayes (1985) although their development and history extends back much further in philosophical literature, see for example Hamblin (1967, 1971). However, despite the intuitive connection that can be drawn between space and time in terms of such logics, until fairly recently, little work in AI has centred on the development and use of interval logics for space.

We describe an interval logic that can be used to reason about space. The similarity of the title with Clarke's (1981) paper 'A calculus of individuals based on 'connection' ', is not accidental. In Randell and Cohn (1989, 1992) and in Randell (1991) we used Clarke's theory as a foundation to build a theory that supported some basic intuitions about the nature of space, time and processes. Although this theory is formally sound, in use, we found some features of both Clarke's theory and our own proved problematic. This led to a re-evaluation of the original theory which is presented below.

The structure of the rest of this paper is as follows. In sections 2 and 3, we give a brief overview of the original theory, and point out the various problems we encountered that led to the development of the revised theory. In section 4 we give the axiomatised theory, drawing out the contrasts with the original theory. In section 5 we discuss the implications of introducing atomic regions into the theory, and in section 6 we discuss related and further work.

2 Overview of the original spatial theory

The original theory (see Randell and Cohn 1989, 1992 and Randell 1991) is based upon Clarke's (1981, 1985) calculus of individuals based on "connection" and is expressed in the many sorted logic LLAMA - see Cohn (1987).

The ontological primitives of the theory include physical objects, regions and other sets of entities. These and other specialisations of these primitive sets of entities are all treated as sorts in the theory and are subsequently embedded in a complete Boolean lattice, forming a sort hierarchy. However, given the scope of this paper, we shall limit the overview of the original theory to that which applies to space, while reminding the reader that what follows is but a small part of a much larger theory.

The basic part of the theory assumes a primitive dyadic relation: $\mathrm{C}(x, y)$ read as 'x connects with y' which is defined on regions; this is axiomatised to be reflexive and symmetric. In terms of points incident in regions, $\mathrm{C}(x, y)$ holds when regions x and y share a common point. Using the relation $\mathrm{C}(x, y)$, a basic set of dyadic relations are defined. These relations describe differing degrees of connection between regions from being disconnected, to being externally connected, allowing partial overlap, one region being a tangential part of the other, or a nontangential part, and so on. All degrees of connection from being externally connected to sharing mutual parts and thus being identical are formally defined.

The theory also supports a set of functions that define the Boolean composition of regions, and a set of topological functions that allow for the explicit representation of the interior, the closure and the exterior of particular regions. We also extend the basic theory outlined by Clarke by including a further set of dyadic relations that are used to describe regions being either inside, partially inside, or outside another. Several variants are defined.

The spatial part of the theory represents but a part of a much larger theory, which is now briefly covered. The theory enables the user to describe states, events and processes. For this a set of ternary relations are introduced that enable one to relate pairs of bodies using the dyadic relations mentioned above over time. These are subsequently used to create a set of envisioning axioms in the general theory, which impose constraints upon the manner in which bodies can vary in their degree of connection over time. These form the basis of processes described in the theory, where processes are described in terms of stipulated sequences of direct topological transitions allowed between sets of objects. These processes can either be reasoned about using a direct theorem proving implementation of the theory, or by using a simulation program - see Cui, Cohn and Randell (1992) for further details.

3 Problems

There are several problems that have arisen during our course of research using the original theory. These can be conveniently classified under three distinct, but related headings: conceptual, pragmatic and computational. We shall discuss these in turn.

A common question asked of us concerning the original theory was why we needed to introduce the topological distinctions between the types of regions assumed by the general theory. From the naive point of view, it seemed odd to have open, semi-open and closed regions as a model for regions. This point simply reflects a general concern made by writers in both philosophy and science, that a remoteness exists between the facts of actual observation and the descriptive language used. In Philosophy, this has resulted in a strong interest in developing languages with a clear primitive observational or phenomenal content; languages that can be directly related to the world around us (Hamblin 1971). For example, in terms of content, it seems odd that two regions can be distinct, but that each occupies the same amount of space, as in the case where we take an open region, and its closure.[1] Moreover, given the explicit use of different types of topological regions for describing space, we have the odd result that if a body maps to a closed region of space (which is a natural association), its complement is open, and that if we consider a body which is broken into two parts, then we have a problem how to split the regions so formed, since any closed interval that is split into two must have a semi-open part, and which is which?[2] From the standpoint of our naive understanding of the world, this topological structure is arguably too rich for our purposes, and in any case appearing in this formal theory, it poses some deep conceptual problems.

Given the choice between two possible theories used for formally representing space, the ease by which a person can understand and use the theory must be taken into account. The basic part of the original theory, concerning regions required the user to be familiar with general topology, both in order to understand the theory, and for any person wishing to extend the theory. We thought this restriction could be eased, but this required a change in the ontology of regions assumed by the original theory, and changes in the extant axiomatisation.

Clarke's (1981, 1985) calculus of individuals is simply presented as an unsorted first order theory, and as such, questions of implementation are understandably not addressed. However, in our case, we had to keep implementational and efficiency questions to the fore. We decided to use a sorted logic, since their effectiveness in reducing the search space for many problems in automated reasoning is well known. Also we wanted to keep our syntax as clear as possible, by absorbing all the monadic predicates in the theory and pushing these into the sortal part of the logic. We originally decided to implement our original theory using Cohn's (1987) sorted logic LLAMA, but this required much groundwork first, since the logic requires the user to first specify the positions of the sorts in the sort hierarchy[3]. This required us to first prove in the sorted theory, for any two potential sorts (being the monadic predicates of the unsorted theory), whether they were disjoint or whether one subsumed the other. This proved to be a particularly difficult and tedious task, which was made especially difficult given the spartan nature of the primitives used in the theory, which meant even basic theorems could prove difficult to tease out. Part of the problem simply lay in the number of potential subsorts, of the sort REGION, we had defined, and again this in part stemmed from the topological basis of the theory stemming from Clarke's theory[4].

[1] A very clear example of this was suggested by Antony Galton, who pointed out that the northern hemisphere, with or without the equator includes the same amount of regional space - the former being a closed region, the latter a semi-open one

[2] It is interesting to note too that the same difficulties for space also arise in the temporal model, for example, deciding whether the order of intervals should be either ( ], or [ ). See Galton (1990) for further discussion.

[3] However, more recently, LLAMA has been relaxed and only partial sort information need be specified – see Cohn(1992).

[4] By having three kinds of regions (open, closed, semi-open), the number of sorts was immediately increased threefold.

Taking all these factors into account we eventually decided to investigate how the theory could be simplified; this is presented below.

4 The new theory

The new theory, like the original theory, is based upon Clarke's calculus of individuals based on "connection" and again is expressed in the many sorted logic LLAMA. Reasons of space mean that we cannot give full details of the sorted logic assumed below. However, for the purposes of reading this paper, all the reader should bear in mind is that LLAMA allows arbitrary ad hoc polymorphism, and that the variables are not explicitly typed, but that their associated sorts are derived implicitly from their argument positions in specified formulae. We will occasionally highlight certain sortal restrictions; in this case sorts in the theory will be indicated by strings of upper case letters, e.g. REGION, SPATIAL and NULL.

The ontological primitives of the (extended) new theory include physical objects, regions and other sets of entities. These and other specialisations of these primitive sets of entities are all treated as sorts in the theory and are subsequently embedded in a complete Boolean lattice, forming a sort hierarchy. However, here, by restricting ourselves to a theory describing space, we shall only concern ourselves with those sorts that specialise the sort SPATIAL.

Regions in the theory support either a spatial or temporal interpretation. Informally, these regions may be thought to be potentially infinite in number, and any degree of connection between them is allowed in the intended model, from external contact to identity in terms of mutually shared parts.

The basic part of the formalism assumes one primitive dyadic relation: $\mathrm{C}(x,y)$ read as 'x connects with y'. For the basic part of the theory, the individuals can be interpreted as either spatial or temporal regions, but as we are describing a theory for space, a spatial interpretation is assumed in the pictorial model we give in Figure 1. The relation $\mathrm{C}(x,y)$ is reflexive and symmetric. We can give a topological model to interpret the theory, namely that $\mathrm{C}(x,y)$ holds when the topological closures of regions x and y share a common point.[5] Two axioms are introduced.

$\forall x \mathrm{C}(x,x)$
$\forall xy[\mathrm{C}(x,y) \rightarrow \mathrm{C}(y,x)]$

Using $\mathrm{C}(x,y)$, a basic set of dyadic relations are defined: '$\mathrm{DC}(x,y)$' ('x is disconnected from y'), '$\mathrm{P}(x,y)$' ('x is a part of y'), '$\mathrm{PP}(x,y)$ ('x is a proper part of y'), '$x = y$' ('x is identical with y'), '$\mathrm{O}(x,y)$' ('x overlaps y'), '$\mathrm{DR}(x,y)$' ('x is discrete from y') '$\mathrm{PO}(x,y)$' ('x partially overlaps y'), '$\mathrm{EC}(x,y)$' ('x is externally connected with y)', '$\mathrm{TPP}(x,y)$' ('x is a tangential proper part of y') and '$\mathrm{NTPP}(x,y)$' ('x is a nontangential proper part of y'). The relations: P,PP,TPP and NTPP being non-symmetrical support inverses. For the inverses we use the notation Φ^{-1}, where $\Phi \in$ {P,PP,TPP and NTPP}. Of the defined relations, DC,EC,PO,=,TPP,NTPP and the inverses for TPP and NTPP are provably mutually exhaustive and pairwise disjoint. The complete set of relations described above can be embedded in a relational lattice. This is given in Figure 1. The symbol $\top$ is interpreted as tautology and the symbol $\perp$ as contradiction. The ordering of these relations is one of subsumption with the weakest (most general) relations connected directly to top and the strongest (most specific) to bottom. For example, TPP implies PP, and PP implies either TPP or NTPP. A greatest lower bound of bottom indicates that the relations are mutually disjoint, for example with TPP and NTPP, and P and DR. This lattice corresponds to a set of theorems (eg. $\forall xy[\mathrm{PP}(x,y) \rightarrow [\mathrm{TPP}(x,y) \vee \mathrm{NTPP}(x,y)]]$) which we have verified.

$\mathrm{DC}(x,y) \equiv_{def} \neg\mathrm{C}(x,y)$
$\mathrm{P}(x,y) \equiv_{def} \forall z[\mathrm{C}(z,x) \rightarrow \mathrm{C}(z,y)]$
$\mathrm{PP}(x,y) \equiv_{def} \mathrm{P}(x,y) \wedge \neg\mathrm{P}(y,x)$
$x = y \equiv_{def} \mathrm{P}(x,y) \wedge \mathrm{P}(y,x)$
$\mathrm{O}(x,y) \equiv_{def} \exists z[\mathrm{P}(z,x) \wedge \mathrm{P}(z,y)]$
$\mathrm{PO}(x,y) \equiv_{def} \mathrm{O}(x,y) \wedge \neg\mathrm{P}(x,y) \wedge \neg\mathrm{P}(y,x)$
$\mathrm{DR}(x,y) \equiv_{def} \neg\mathrm{O}(x,y)$
$\mathrm{TPP}(x,y) \equiv_{def} \mathrm{PP}(x,y) \wedge \exists z[\mathrm{EC}(z,x) \wedge \mathrm{EC}(z,y)]$
$\mathrm{EC}(x,y) \equiv_{def} \mathrm{C}(x,y) \wedge \neg\mathrm{O}(x,y)$
$\mathrm{NTPP}(x,y) \equiv_{def} \mathrm{PP}(x,y) \wedge \neg\exists z[\mathrm{EC}(z,x) \wedge \mathrm{EC}(z,y)]$
$\mathrm{P}^{-1}(x,y) \equiv_{def} \mathrm{P}(y,x)$
$\mathrm{PP}^{-1}(x,y) \equiv_{def} \mathrm{PP}(y,x)$
$\mathrm{TPP}^{-1}(x,y) \equiv_{def} \mathrm{TPP}(y,x)$
$\mathrm{NTPP}^{-1}(x,y) \equiv_{def} \mathrm{NTPP}(y,x)$

In the original theory, several other defined relations (missing here) were defined. These were the set of relations: '$\mathrm{TP}(x,y)$' ('x is a tangential part of y'), '$\mathrm{NTP}(x,y)$' ('x is a nontangential part of y'),'$\mathrm{TPI}(x,y)$' ('x is the identity tangential part of y'), and, '$\mathrm{NTPI}(x,y)$' ('x is the identity nontangential part of y'). We also omit in the new theory the set of topological functions introduced by Clarke, and adopted by us in the original theory. In this revised theory, we make no formal distinction in our model between open, semi-open and closed regions used to interpret this part of the formalism (as was done in the original theory), so for example now the identity relation does not split into two specialisations here, as it did in the original theory to account for the differences between types of regions. A similar rationale applies for the explicit introduction of the tangential part and nontan-

[5] In Clarke's theory and in our original theory, when two regions x and y connect, they are said to share a point in common; thus the interpretation of the connects relation in the new theory is weaker.

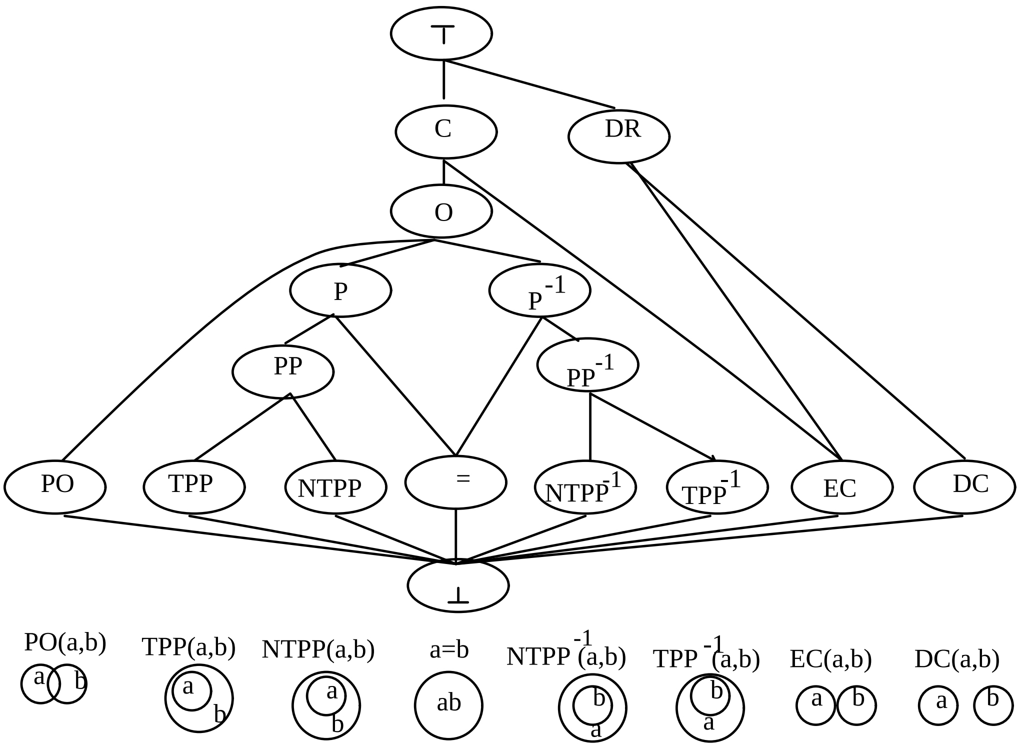

Figure 1: A lattice defining the subsumption hierarchy of the dyadic relations defined solely in terms of the primitive relation $\mathrm{C}(x, y)$.

gential part relations mentioned above - see Randell (1991), Randell and Cohn (1989), Randell and Cohn (1992) for further details.

Excepting the definition for the complement of a region, the Boolean part of the new theory follows the original theory, and Clarke's. The Boolean functions[6] are: '$\mathrm{sum}(x, y)$' which is read as 'the sum of x and y', 'Us' as 'the universal (spatial) region', '$\mathrm{compl}(x)$' as 'the complement of x', '$\mathrm{prod}(x, y)$' as 'the product (i.e. the intersection of x and y' and '$\mathrm{diff}(x, y)$' as 'the difference of x and y'. The functions: '$\mathrm{compl}(x)$', '$\mathrm{prod}(x, y)$' and '$\mathrm{diff}(x, y)$' are partial but are made total in the sorted logic by simply specifying sorts restrictions and by introducing a new sort called NULL. The sorts NULL and REGION are disjoint.

$$
\begin{aligned}
&\mathrm{sum}(x, y) =_{def} \iota y[\forall z[\mathrm{C}(z, y) \leftrightarrow [\mathrm{C}(z, x) \vee \mathrm{C}(z, y)]]] \\
&\mathrm{compl}(x) =_{def} \iota y[\forall z[[\mathrm{C}(z, y) \leftrightarrow \neg\mathrm{NTPP}(z, x)] \wedge \\
&\qquad\qquad [\mathrm{O}(z, y) \leftrightarrow \neg\mathrm{P}(z, x)]]] \\
&\mathrm{Us} =_{def} \iota y[\forall z[\mathrm{C}(z, y)]] \\
&\mathrm{prod}(x, y) =_{def} \iota z[\forall u[\mathrm{C}(u, z) \leftrightarrow \\
&\qquad\qquad \exists v[\mathrm{P}(v, x) \wedge \mathrm{P}(v, y) \wedge \mathrm{C}(u, v)]]] \\
&\mathrm{diff}(x, y) =_{def} \iota w[\forall z[\mathrm{C}(z, w) \leftrightarrow \\
&\qquad\qquad \mathrm{C}(z, \mathrm{prod}(x, \mathrm{compl}(y)))]] \\
&\forall xy[\mathrm{NULL}(\mathrm{prod}(x, y)) \leftrightarrow \mathrm{DR}(x, y)]
\end{aligned}
$$

In Clarke (1981, 1985) (and also in our original theory) the complement definition is defined so that a region y connects with the complement of region x if and only if y is not a part of x. This has the formal consequence that no region is connected with its own complement.[7] However, this result is not formally derivable in the new theory, and moreover must not be so given the new interpretation. This arises from the new interpretation for the connects relation, since every region (which is

[6] $\alpha(\overline{x}) =_{def} \iota y[\Phi[\alpha(\overline{y})]$ means $\forall \overline{x}[\Phi(\alpha(\overline{x}))]$; thus, e.g., the definition for $\mathrm{prod}(x, y)$ is translated out (in the object language) as: $\forall xyz[\mathrm{C}(z, \mathrm{prod}(x, y)) \leftrightarrow \exists w[\mathrm{P}(w, x) \wedge \mathrm{P}(w, y) \wedge \mathrm{C}(z, w)]]$.

[7] Here we are assuming certain restrictions on x. In the unsorted theory assumed by Clarke, this amounts to x not being identical to the universal region - our constant Us.

not identical to the universal region) will be connected with its own complement. In fact this difference is reflected in the theorem: $\forall x \mathrm{EC}(x, \mathrm{compl}(x))$ which contradicts the related theorem described above.

An additional axiom is then added to the new theory which stipulates that every region has a nontangential proper part:

$$\forall x \exists y [\mathrm{NTPP}(y, x)] \tag{i}$$

This axiom mirrors a formal property of Clarkes' theory, where he stipulates that every region has a nontangential part, and thus an interior (remembering that in Clarke's theory a topological interpretation is assumed).

4.1 One piece regions

Clarke's theory supports a model where regions may topologically connected (i.e. in one piece) or disconnected (in more than one piece). A definition for a connected region is given - which states that a region is disconnected iff it cannot be split into two disjoint parts. The same type of model supporting either individual connected or disconnected regions appears in the new theory, only here the definition for an individual connected region does not need to incorporate the distinction between topological types of regions, i.e. being open, semi-open or closed. The definition simply states that an individual region is connected if it cannot be split into parts whose union is that region, and where these parts are not connected to each other[8], i.e.

$$\mathrm{CON}(x) \equiv_{def} \forall yz[\mathrm{sum}(y, z) = x \rightarrow \mathrm{C}(y, z)]$$

4.2 Proper and Improper regions

A proper region is defined to be a region that has a nontangential proper part, and an improper region, a region that is not a proper region.

$$\text{PROP-REGION}(x) \equiv_{def} \exists z \mathrm{NTPP}(z, x)$$

In the basic theory, where we allow space to be continuously decomposed into a set of nontangential proper parts, every region becomes a proper region, and no region an improper region. However, in section 5 we discuss the possibility of adding atoms into the formal theory, and by positing atoms, improper regions can be defined. Examples of improper regions would be single atoms, and various clusters of atoms forming strings, rings, and sheets in 3-space. As the possibility of defining these objects requires atoms to be posited, and that the question of whether or not atoms can be included is a complex one, we refer the reader to section 5 where this matter is discussed in more detail.

[8] The original definition in Randell and Cohn (1989) had to be modified since it referred to the closure of a region.

4.3 Inclusion vs Containment

As with the original theory (but missing in Clarke) a primitive function 'conv(x)' ('the convex-hull of x') is defined and axiomatised. We assume here that conv is only well sorted when defined on one piece regions.

$$\forall x \mathrm{P}(x, \mathrm{conv}(x))$$
$$\forall x \mathrm{P}(\mathrm{conv}(\mathrm{conv}(x)), \mathrm{conv}(x))$$
$$\forall x \forall y \forall z[[\mathrm{P}(x, \mathrm{conv}(y) \wedge \mathrm{P}(y, \mathrm{conv}(z))] \rightarrow \mathrm{P}(x, \mathrm{conv}(z))]$$
$$\forall x \forall y[[\mathrm{P}(x, \mathrm{conv}(y)) \wedge \mathrm{P}(y, \mathrm{conv}(x))] \rightarrow \mathrm{O}(x, y)]$$
$$\forall x \forall y[[\mathrm{DR}(x, \mathrm{conv}(y)) \wedge \mathrm{DR}(y, \mathrm{conv}(x))] \leftrightarrow \mathrm{DR}(\mathrm{conv}(x), \mathrm{conv}(y))]$$

We use this function to define a set of relations which describe regions being inside, partially inside and outside, e.g. 'INSIDE(x, y)' (x is inside y'), 'P-INSIDE(x, y)' ('x is partially inside y') and 'OUTSIDE(x, y)' ('x is outside y'). This particular set of relations extends below DR(x, y) in the basic theory. The developed theory actually supports many specialisations of these particular relations, with, for example, one region being wholly outside, or just outside, or just inside, or wholly inside another - see Randell and Cohn (1989, 1992) and Randell (1991). However, here we restrict the set of defined relations to the specialisations given above, their inverses, and the set of relations that result from non-empty intersections. The set of base relations for this particular set are finally generated by defining a further set of specialisations of these relations using the EC and DC relations. In the interest of space, only a subset of the constructible defined relations are given below. However the interested reader should have no difficulties actually generating the formal definitions from the schema given below.

Here are the formal definitions for the named relations introduced above, together with their inverses:

$$\mathrm{INSIDE}(x, y) \equiv_{def} \mathrm{DR}(x, y) \wedge \mathrm{P}(x, \mathrm{conv}(y))$$
$$\text{P-INSIDE}(x, y) \equiv_{def} \mathrm{DR}(x, y) \wedge \mathrm{PO}(x, \mathrm{conv}(y))$$
$$\mathrm{OUTSIDE}(x, y) \equiv_{def} \mathrm{DR}(x, \mathrm{conv}(y))$$
$$\mathrm{INSIDE}^{-1}(x, y) \equiv_{def} \mathrm{INSIDE}(y, x)$$
$$\text{P-INSIDE}^{-1}(x, y) \equiv_{def} \text{P-INSIDE}(y, x)$$
$$\mathrm{OUTSIDE}^{-1}(x, y) \equiv_{def} \mathrm{OUTSIDE}(y, x)$$

A new set of base relations (using the relations defined immediately above) are constructed according to the following schema:

$$\alpha\_\beta(x, y) \equiv_{def} \alpha(x, y) \wedge \beta(x, y)$$

where: $\alpha \in$ {INSIDE, P-INSIDE, OUTSIDE}, and $\beta \in$ {INSIDE^{-1}, P-INSIDE^{-1}, OUTSIDE^{-1}}, excepting where α = INSIDE and β =INSIDE^{-1} Each of these 'composite' relations then split into two variants, the case where x and y EC, and the case where they DC. This finally gives rise to the new set of base relations in the extended theory, which now number 22 instead of 8 in the revised basic theory (cf. 23 and 9 in the original theory).

Two functions capturing the concept of the inside and the outside of a particular region are also definable (where 'inside(x)' is read as 'the inside of x', and 'outside(x)' as 'the outside of x' respectively:

$$\mathrm{inside}(x) =_{def} \iota y[\forall z[\mathrm{C}(z,y) \leftrightarrow \exists w[\mathrm{INSIDE}(w,x) \wedge \mathrm{C}(z,w)]]]$$
$$\mathrm{outside}(x) =_{def} \iota y[\forall z[\mathrm{C}(z,y) \leftrightarrow \exists w[\mathrm{OUTSIDE}(w,x) \wedge \mathrm{C}(z,w)]]$$

4.4 Geometrically Inside vs Topologically Inside

In the previous section the DR relation is specialised to cover relations describing objects being either inside, partially inside or outside other objects. However this ignores some useful distinctions that can be drawn between different cases of bodies being inside another. In this case we separate out the case where one body is topologically inside another, and where one body is inside another but not topologically inside – this we call being geometrically inside. The important point of one body being topologically inside another is that one has to 'cut' through the surrounding body in order to reach and make contact with the contained body. In the geometrical variant this is not the case.

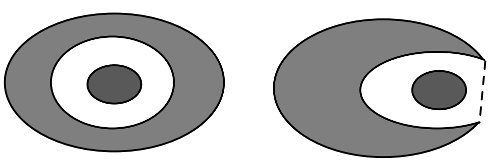

Figure 2: The distinction between being topologically and geometrically inside. The dashed lines appearing here and in Figure 3 indicate the extent of the convex hull of the surrounding bodies.

$$\mathrm{TOP\text{-}INSIDE}(x,y) \equiv_{def} \mathrm{INSIDE}(x,y) \wedge \forall z[[\mathrm{CON}(z) \wedge \mathrm{C}(z,x) \wedge \mathrm{C}(z,\mathrm{outside}(y)] \rightarrow \mathrm{O}(z,y)]$$
$$\mathrm{GEO\text{-}INSIDE}(x,y) \equiv_{def} \mathrm{INSIDE}(x,y) \wedge \neg\mathrm{TOP\text{-}INSIDE}(x,y)$$

It is also possible to specialise the relation of being geometrically inside – in this case setting up definitions to distinguish between the following pictorial representations – Figure 3:

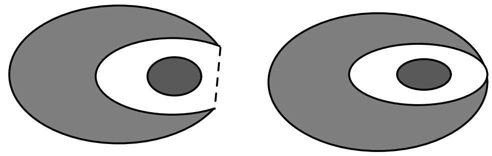

Figure 3: Two variants of being geometrically inside. In the right hand figure the two 'arms' meet at a point.

In order to make this formal distinction we first set up a stronger case of a connected or one-piece region to that assumed above. The important part of the following definition is the $\mathrm{P}(\mathrm{conv}(\mathrm{sum}(v,w)),x)$ literal in the consequent of the definiens. This condition ensures that the connection between any two parts of a region whose sum equals that region, is not point or edge connected. That is to say it ensures a 'channel' region exists connecting any two connected parts. This notion of being connected mirrors and simplifies our previous definition of a quasi-manifold – in this case we use the concept of a convex body rather than use topological and Boolean concepts in the earlier definition – see Randell and Cohn (1989).

$$\mathrm{CON}'(x) \equiv_{def} \mathrm{CON}(x) \wedge \forall yz[[\mathrm{sum}(y,z) = x \rightarrow \mathrm{C}(y,z)] \rightarrow \exists vw[\mathrm{P}(v,y) \wedge \mathrm{P}(w,z) \wedge \mathrm{P}(\mathrm{conv}(\mathrm{sum}(v,w)),x)]]$$

Now we give the formal distinction between the two cases of being geometrical inside. In the first case a 'channel' region exists connecting the outside of the surrounding body with the contained body, in the second case the surrounding body has closed forming (in this case) a point connection. In both cases we can see how in contrast with the notion of being topologically inside, it is possible to construct a line segment that connects with both the surrounding body and the contained body without cutting through the surrounding body. Definitions distinguishing between the two cases are as follows, where the open and closed variants respectively refer to the first and second cases described above.

$$\mathrm{GEO\text{-}INSIDE\text{-}OPEN}(x,y) \equiv_{def} \mathrm{GEO\text{-}INSIDE}(x,y) \wedge \mathrm{CON}'(\mathrm{sum}(\mathrm{inside}(y),\mathrm{outside}(y)))$$
$$\mathrm{GEO\text{-}INSIDE\text{-}CLOSED}(x,y) \equiv_{def} \mathrm{GEO\text{-}INSIDE}(x,y) \wedge \mathrm{CON}(\mathrm{sum}(\mathrm{inside}(y),\mathrm{outside}(y))) \wedge \neg\mathrm{CON}'(\mathrm{sum}(\mathrm{inside}(y),\mathrm{outside}(y)))$$

4.5 Theorems in the new theory

As mentioned above, some important differences exist between both Clarke's and the original theory, and the new theory. For brevity we shall subsume our original theory under Clarke's, when making the contrast. Where a difference arises between some theorem of Clarke's and our own original theory, we shall make this explicit. First we demonstrate how the topological distinction drawn between open, semi-open and closed regions sanctioned in Clarke's theory cannot be made in the new theory. For Clarke, two regions x and y are identical iff any region connecting with x connects with y and vice-versa, i.e. $\forall xy[x = y \leftrightarrow$

$\forall z[\mathrm{C}(z,x) \leftrightarrow \mathrm{C}(z,y)]]$; however in the new theory, an additional theorem concerning identity becomes provable which is not a theorem in Clarke's theory. This is: $\forall xy[x = y \leftrightarrow \forall z[\mathrm{O}(z,x) \leftrightarrow \mathrm{O}(z,y)]]$. The topological model used in Clarke's theory, together with the absence of boundary elements as regions, explains why this formula is not derivable. For example, given the closure of region x and its interior, then any region overlapping the closure of x, overlaps the interior of x, and vice-versa, (remembering that overlapping regions entail that they share a common interior point) but from this we cannot allow the interior of x to be identical with its closure, which would follow if the related formula were to be a theorem in Clarke's theory.

The next important difference between Clarke's and the new theory is the formula: $\forall xy[\mathrm{PP}(x,y) \rightarrow \exists z[\mathrm{P}(z,y) \wedge \neg\mathrm{O}(z,x)]]$ which is provable in the new theory, but not in Clarke's. Given Clarke's theory supports open, semi-open and closed regions as a model, it becomes clear why this formula is not provable in Clarke's theory, since while the interior of a region is a proper part of its closure, (and boundaries are not regions) there is no other part of the closure which does not overlap the interior. If one adds the condition that the regions in question are closed, then, the formula is true of Clarke's theory, but this condition is waivered in the new theory. Another related formula is: $\forall xy[\mathrm{PO}(x,y) \rightarrow [\exists z[\mathrm{P}(z,y) \wedge \neg\mathrm{O}(z,x)] \wedge \exists w[\mathrm{P}(w,x) \wedge \neg\mathrm{O}(w,y)]]]$, which is a theorem in the new theory but not in Clarke's. A counter example arises in Clarke's theory where we have two semi-open spherical regions, x and y (with identical radii), such that the northern hemisphere of x is open and the southern hemisphere is closed, and the northern hemisphere of y is closed and the southern hemisphere open. If x and y are superimposed so that their centres and equators coincide, then x and y will partially overlap, but no part of x is discrete from y, and vice-versa. Both these theorems in the new theory show that a positive Boolean difference exists between y and x when x is a proper part of y. Again in Clarke's theory this result only follows when both x and y are closed regions.

In the new theory, $\forall x\, \mathrm{EC}(x, \mathrm{compl}(x))$ holds; this contrasts with the theorem: $\forall x\, \mathrm{DC}(x, \mathrm{compl}(x))$ in both the original and in Clarkes' theories. Also here it is worth pointing out that in the original theory (which included Clarke's set of topological operators) we included the axiom: $\forall x \mathrm{EC}(\mathrm{cl}(x), \mathrm{cl}(\mathrm{compl}(x)))$ which ensured that the closure of x externally connected with the closure of the complement of x, where x was restricted so that it was not the universal region.[9]

Other interesting theorems are: $\forall xyz[[\mathrm{C}(z,y) \wedge \neg\mathrm{C}(z,x)] \rightarrow \exists w[\mathrm{P}(w,y) \wedge \neg\mathrm{O}(w,x) \wedge \mathrm{C}(z,w)]]$, and $\forall xy[[\mathrm{PP}(x,y) \wedge \mathrm{Connected}(y)] \rightarrow \exists z[\mathrm{P}(z,y) \wedge \mathrm{EC}(z,x)]]$. Note for the latter formula to be a theorem, an additional restriction on variable y is required, namely that y is a place-holder for a one-piece region.

[9] It turns out that if we assume the universal region is topologically connected (which is a definable concept in Clarkes' theory) and that x is not the universal region, we can prove this as a theorem. We are indebted to Laure Vieu who demonstrated the proof to us.

Readers familiar with either Clarke's theory, or our own original theory may be wondering what happens to the relations TP and NTP which are excluded here. In the new theory, we find that if we defined these relations and added them to the extant set, the two relations would give rise (on the assumption that x is not identical with the universal spatial region) to the theorems: $\forall x \mathrm{TP}(x,x)$ and $\forall x \neg\mathrm{NTP}(x,x)$ respectively. The latter indicates that no positive instance of the relation NTP which is not a case of NTPP can arise in any model of the new theory. Thus we omit NTP and TP for reasons of symmetry and neither relation appears in the relational lattice depicted in Figure 1.

4.6 Transitivity Tables for the new theory

A transitivity table is defined as follows. Given a particular theory Σ supporting a set of mutually exhaustive and pairwise disjoint dyadic relations, three individuals, a, b and c and a pair of dyadic relations R_1 and R_2 selected from Σ such that $R_1(a,b)$ and $R_2(b,c)$, the transitive closure $R_3(a,c)$ represents a disjun ction of all the possible dyadic relations holding between a and c in Σ. Each $R_3(a,c)$ result can b e represented as one entry of a matrix for each $R_1(a,b)$ and $R_2(b,c)$ ordered pair. If there are n dyadic relations supported by Σ, then there will be $n \times n$ entries in the matrix. T his matrix is called a transitivity table. A well known example of a transitivity table appears in an implementation of Allen's temporal logic (Allen 1983); we also give a transitivity table for our original theory in (Randell, Cohn and Cui 1992) and in (Randell and Cohn 1992).

The new transitivity table is essentially the same as the original one excepting that the new matrix for the basic set of base relations has only one relation covering the identity relation, and not two as before. The new table is easily constructed by simply eliminating the row and column labelled NTPI, eliminating every NTPI entry which appears in each cell, and replacing TPI with =. The table is an 8×8 matrix (64 cells) averaging ≈ 3 entries per cell. For the basic extension to this table (including the inside, partially inside and outside relations) the matrix increases to 22×22 (484 cells) averaging ≈ 9 entries per cell. On these two examples, the increase in the number of base relations does not appear to increase the complexity of the number of entries in the cells generated. The extended transitivity table furhter increases to 30×30 (900 cells) with the specialisation of the inside relation covering the distinction between being topologically inside and being geometrically inside. Note that this does not exhaust the maximal number of base rela-

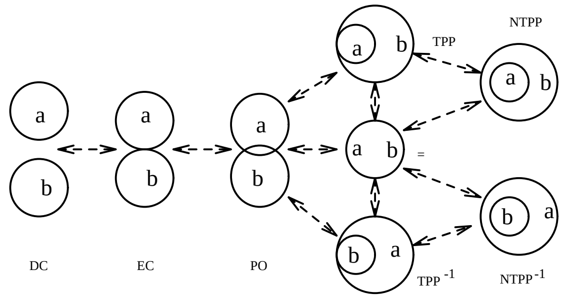

Figure 4: A pictorial representation of the base relations and their direct topological transitions.

tions that can be generated from the definitions given in this paper – for we have not taken into account the distinction made between the two defined cases of one region being geometrically inside another.

As each cell in a transitivity table corresponds to a theorem, computing these large transitivity tables is a non-trivial task – see Randell, Cohn and Cui (1992). We have recently simplified this task by using a program that uses a bit-string model to generate all possible transitivity table configurations for a given set of base relations. The original 9×9 table has been formally proved, and the program constructed to generate the larger 22×22 table conforms with the predicted entries for the 8×8 table.[10]

4.7 Envisioning axioms in the new theory

As mentioned above, we express different sets of base relations in the form of a set of envisioning axioms. These stipulate direct transitions that are allowed between pairs of objects over time. A pictorial representation of the basic set of base relations and their direct topological transitions in the new theory is given in Figure 4.[11]

For the basic set of base relations - the set DC,EC,PO,=, TPP,NTPP and the inverses for TPP and NTPP, no practical difference arises from that used in the original theory (using the set of 9 base relations). This arises simply because in the domains we modelled we simply mapped the named individuals to closed regions, thus eliminating the base relation NTPI which is only true for open regions. However, if we add NTPI into the envisioning axioms, then the number of paths connecting nodes in the graph for the 9 base relations compared with that for the graph generated from the new theory (with 8 base relations) reduces from 17 to 11.[12]

4.8 Models and structures for the new theory

We have already given one model for the new theory, interpreting the C relation in terms of two regions whose closures share a common point. However, other models exist. We could simply state that two regions connect, when it is not possible to ‘fit’ another distinct region between the two, or alternatively to say when the distance between them is zero. Clarke (1981) only suggests the point based topological interpretation as one possible interpretation for his axiomatisation.

For the new theory there is an important metatheoretic restriction concerning Boolean sums or unions of regions, namely that infinite unions cannot be allowed. If infinite unions are allowed, the theory becomes inconsistent. The proof sketch is as follows. In the theory we have an axiom that ensures every region has a nontangential proper part, and a theorem that states that if region x is a proper part of region y, then there exists another region z that is part of y, but is disjoint with x. From both of these, it follows that every region is subdivided into an infinite set of nontangential proper parts. However if we take the infinite union of all the nontangential proper parts of y, then in the limit this union becomes identical with y. However, the definition for NTPP requires no region to externally connect with y, but y now identical with the infinite union of all its nontangential proper parts, must externally connect with its own complement – which is inconsistent. Viewed another way this result simply illustrates the fact that (on pain of contradiction) interiors (in the topological sense of the term) cannot be explicitly introduced into the theory. In fact it can be shown that by adding the definition (for the interior of region x):

$$\mathrm{int}(x) =_{def} \iota y[\forall z[\mathrm{C}(z, y) \leftrightarrow \exists w[\mathrm{NTPP}(w, x) \wedge \mathrm{C}(z, w)]]$$

and positing the existence of interiors, a formal contradiction is generated.

4.9 Comparisons with the Classical Calculus of Individuals

Readers familiar with Leonard and Goodman’s (1940) (classical) calculus of individuals will notice similar-

[10] The generation of the 22×22 table took 2 days CPU time on a Sun Sparc IPC.

[11] Note that here as in the network used in the original theory, we have assumed that the regions depicted have nontangential proper parts. This means no direct transition from e.g. EC to identity is allowed which would arise if both regions were atomic.

[12] This assumes that a legitimate path connects the node NTPI with the nodes TPI,PO,TPP,NTPP and the inverses for TPP and NTPP. Ontologically speaking this is the most liberal result where we allow regions to change their topological type over time, i.e. from non-open to open as in for example the path linking TPI and NTPI. Other less liberal linkages may well be envisaged which would reduce the number of connections between nodes.

ities between this calculus and the new calculus described above. In the classical calculus, DR is axiomatised to be irreflexive and symmetrical, and is used to create a set of dyadic relations and Boolean operators defined on individuals. No analogues of DC and EC (defined in Clarke's calculus) are defined in the classical calculus. With the weaker relation C this distinction can be made. The new theory contains, as part of its complement definition, a conjunct that mirrors the definition for complementation in the classical calculus, i.e. the formula: $\forall xy[\mathrm{O}(x, \mathrm{compl}(y)) \leftrightarrow \neg\mathrm{P}(x, y)]$. This conjunct forces the following formula: $\forall xy[\mathrm{P}(x, y) \leftrightarrow \forall z[\mathrm{O}(z, x) \rightarrow \mathrm{O}(z, y)]$ to be a theorem in the new theory; in fact this equivalence mirrors the definition for P in the classical calculus, where P is defined solely in terms of O. The new theory straddles between Clarke's and the classical calculi of individuals.

5 Atomic regions: a discussion

In Randell, Cui and Cohn (1992) we allowed atomic regions or atoms to be introduced into the ontology. Atoms were defined as regions with no proper parts, and an existential axiom was added that ensured every region had an atom as a part. In the intended model, atoms were understood to be 'very small' regions. Atoms were then used in the definition of what we called the skin of a region. This skin is comparable to the notion of a mathematical surface, except that unlike a surface proper, the skin of a region was understood to have non-zero thickness. The definition for the skin of a region simplified the analogous definition given in Randell (1991), and in Randell and Cohn (1992); and was a direct result of our new theory.

As the basic theory supports a model with a continuous decomposition of regions into nontangential proper parts (being a direct consequence of axiom (i) above), some restriction was necessary to avoid building inconsistency into the theory. Axiom (ii) used below consequently replaced axiom (i). The basic extension was presented as follows:

$$\mathrm{ATOM}(x) \equiv_{def} \forall y[\mathrm{P}(y, x) \rightarrow y = x]$$
$$\forall x[\neg\mathrm{ATOM}(x) \rightarrow \exists y[\mathrm{ATOM}(y) \wedge \mathrm{P}(y, x)]]$$
$$\forall x[\neg\mathrm{ATOM}(x) \rightarrow \exists y[\mathrm{NTPP}(y, x)]] \qquad \text{(ii)}$$
$$\mathrm{skin}(x) =_{def} \iota y[\forall z[\mathrm{C}(z, y) \leftrightarrow \exists v[\mathrm{ATOM}(v) \wedge \mathrm{TPP}(v, x) \wedge \mathrm{C}(z, v)]]]$$

One improvement to the above can immediately be made: axiom (ii) is clearly too restrictive – it should be rewritten as

$$\forall x[\neg\text{PROP-REGION}(x) \rightarrow \exists y[\mathrm{NTPP}(y, x)]]$$

but this is just one half of the definition of PROP-REGION and so is logically redundant! Thus axiom (i) should simply be deleted and not replaced with anything.

5.1 Small is not beautiful: problems posed by atoms

Unfortunately, it turns out that even given the restriction imposed by axiom (ii) above, this is not sufficient to stop inconsistency arising in the atomic variant of this theory. The discovery of this came as something of a surprise, since the axioms and definitions used seemed intuitively correct, until the discovery of the contradiction forced us to look deeper into the axiomatisation. The proof is as follows. Assume an arbitrary atom, call it b. Assume b is not identical to the universe, then b has a complement, and EC(b,compl(b)) follows. However, since b is an atom, then everything connected to b, connects with compl(b). From the definition of P, P(b,compl(b)) also follows. P(b,compl(b)) implies O(b,compl(b)), which implies ¬EC(b,compl(b)). Thus EC(b,compl(b)) and ¬EC(b,compl(b)) – contradiction – QED. The problem lies with the definition of P, which (in the intended domain) is false for atoms, for it is not true that just because every region connected to an atom is connected to its complement, that atom is necessarily part of its complement.

Now it turns out that Clarke's theory is immune from this problem for the following reasons: (i) his definition for complement (being different) ensures that -C(x,compl(x)) follows, and (ii) because of the existence of interiors in his theory. This latter feature ensures the existence of some region that (connects with itself but) does not connect with its complement. And thus it does not follow that an atom posited in Clarke's theory (being identical to its own interior) is part of its own complement. The explicit introduction of (topological) interiors into Clarke's calculus ensures his theory is (at least on this point) sound, but given we sought to eliminate this explicit characterisation of open, semi-open and closed regions other solutions must be sought.

5.2 Small is not beautiful: solutions

Below we give three potential solutions to the problem posed by admitting atoms into the domain. Two of these require that atoms be introduced as a primitive sort, while the third keeps atoms as a definable sort but introduces points. This section covers work in progress, so the proposed solutions must viewed in this light.

The first solution is to make atoms a primitive sort, that is to say we do not give a formal definition for atom as above. Atoms are then allowed to have regions as parts which we call particles, but with the restriction that particles always occur in atoms and do not appear in non-atomic regions without also being embedded in atoms. Thus what we call atoms here are really pseudo-atoms since they contain proper parts. The idea is that for most practical modelling purposes (pseudo) atoms are considered to be the most primitive entity that is explicitly referred to. The contradiction arising from positing atoms now dissolves. Because atoms now have proper parts, this means that it is no longer true that every region which connects with an atom connects with its complement.

Given atoms are represented as a primitive sort, we need to axiomatise their properties. First we stipulate that any two atoms that overlap become identical. Next we add the definition for a particle, together with an axiom that every atom has a particle as a proper part:

$$\forall xy[[\mathrm{ATOM}(x) \wedge \mathrm{ATOM}(y) \wedge \mathrm{O}(x,y)] \rightarrow x = y]$$
$$\mathrm{PARTICLE}(x) \equiv_{def} \exists y[\mathrm{ATOM}(y) \wedge \mathrm{PP}(x,y)]$$
$$\forall x[[\mathrm{ATOM}(x) \rightarrow \exists y[\mathrm{PARTICLE}(y) \wedge \mathrm{PP}(y,x)]$$
$$\forall x[[\neg\mathrm{ATOM}(x) \wedge \neg\mathrm{PARTICLE}(x)] \rightarrow \exists y[\mathrm{P}(y,x) \wedge \mathrm{ATOM}(y)]]$$

The second solution takes atoms and the summation operator as a primitive sort and function respectively and then defines the part/whole relation in terms of summed regions. First we axiomatise C as before and define C on atoms. Then we define the relations DC, = and EC for atoms. (Here it is useful to remember that atoms can only be disconnected, externally connected or be identical.) Axioms defining the standard properties of the summation operator are then given, together with an axiom that ensures that if two atoms are disjoint, their sum is not an atom:

$$\mathrm{DC}(x,y) \equiv_{def} \neg\mathrm{C}(x,y)$$
$$x = y \equiv_{def} \forall z[\mathrm{C}(z,x) \leftrightarrow \mathrm{C}(z,y)]$$
$$\mathrm{EC}(x,y) \equiv_{def} \mathrm{C}(x,y) \wedge \neg(x = y)$$

$$\forall x \mathrm{sum}(x,x) = x$$
$$\forall xy[\mathrm{sum}(x,y) = \mathrm{sum}(y,x)]$$
$$\forall xyz[\mathrm{sum}(x,\mathrm{sum}(y,z)) = \mathrm{sum}(\mathrm{sum}(x,y),z)]$$
$$\forall xy[\neg(x = y) \rightarrow \neg\mathrm{ATOM}(\mathrm{sum}(x,y))]$$
$$\mathrm{REGION}(x) \equiv_{def} \forall y[\mathrm{C}(y,x) \leftrightarrow \exists z[\mathrm{ATOM}(z) \wedge \mathrm{P}(z,x) \wedge \mathrm{C}(y,z)]]$$

Note that for the first group of formulae presented immediately above, all the variables are of sort ATOM; this restriction is relaxed in the second group where all the variables are of sort SPATIAL (remembering that ATOM is a subsort of SPATIAL in this theory).

Next we start to define the set of binary relations which are true for non-atomic regions:

$$\mathrm{P}(x,y) \equiv_{def} \exists z[y = \mathrm{sum}(x,z)]$$
$$\mathrm{O}(x,y) \equiv_{def} \exists z[\mathrm{P}(z,x) \wedge \mathrm{P}(z,y)]$$
$$\mathrm{EC}(x,y) \equiv_{def} \neg\mathrm{O}(x,y) \wedge \exists zu[\mathrm{ATOM}(z) \wedge \mathrm{ATOM}(u) \wedge \mathrm{P}(z,x) \wedge \mathrm{P}(u,y) \wedge \mathrm{EC}(z,u)]$$

The reader should now be able to complete the set of binary relations defined on non-atomic regions, using the earlier set of definitions as a guide. The rest of the axiomatisation then follows as before, excepting of course that the summation operator does not now appear as a definition.

The third solution keeps atoms as a defined sort, but also introduces points as a new primitive sort into the ontology. The general idea is to rework the definition of the part/whole relation in terms of points instead of regions and connection as before.

First the new sort POINT is stipulated to be pairwise disjoint with REGION and NULL. A new primitive relation '$\mathrm{IN}(x,y)$' read as '(point) x is incident in (region) y' is then added; this replaces the primitive C relation used above. IN is axiomatised to be irreflexive and asymmetrical[13]. Then we define both the C and P relation in terms of points, instead of regions as before:

$$\forall x \neg\mathrm{IN}(x,x)$$
$$\forall xy[\mathrm{IN}(x,y) \rightarrow \neg\mathrm{IN}(y,x)]$$
$$\mathrm{C}(x,y) \equiv_{def} \exists z[\mathrm{IN}(z,x) \wedge \mathrm{IN}(z,y)]$$
$$\mathrm{P}(x,y) \equiv_{def} \forall z[\mathrm{IN}(z,x) \rightarrow \mathrm{IN}(z,y)]$$

The crucial point(!) here is that the formula: $\forall xy[\mathrm{P}(x,y) \leftrightarrow \forall z[\mathrm{C}(z,x) \rightarrow \mathrm{C}(z,y)]]$ is now not provable; this serves to block the proof which generated the contradiction described above.

The rest of the axiomatisation then follows that given in the main body of this paper. Note here that we have chosen to replace C with IN as the primitive dyadic relation upon which this axiomatisation is built. It is certainly possible to axiomatise C as before and then axiomatise IN in terms of C and P, i.e. stipulating that two regions connect iff they share a common incident point, and stipulating that one region is part of another iff every point incident in the former is incident in the latter. Our choice is simply based on ontological parsimony, for while connection can be defined in terms of incidence, incidence cannot be defined in terms of connection.

6 Related and Further Work

We have already mentioned Clarke's calculus of individuals, our earlier work of which this present theory is a simplification, and Allen's and Hamblin's work on interval logics. The only other work of which we are aware, that uses Clarke's theory for describing space, is Aurnague (1991) and Vieu (1990). Other work on the description of space using a body rather than a point based ontology, can be found in Laguna (1922), Tarski(1956) and Whitehead (1978). There have been some attempts in the qualitative spatial reasoning literature to employ Allen's interval logic, for describing

[13] Note that the two axioms for IN are not required in a sorted logic.

space, see for example Freksa (1990) and Hernandez (1990), but here a stronger primitive relation used, which does not allow the full range of topological relationships to be formally described as given in both Clarkes' and our original and new theories. Apart from the question raised by adding atoms to the theory, we are currently working on the question as to whether the new theory supports decidable subsets. We have already indicated some extensions to this new logic above, including a temporal extension and extending the ontology further to be able to reason about bodies and describe, states, events and processes. For other extensions to the spatial theory itself, work described in Randell (1991) can also be included. For example, we could add a metric extension to the theory, using either a distance function, or alternatively by adding a ternary relation (along the lines of Van Benthem 1982, appendix A) that gives comparative distances between objects.

7 Conclusions

The new theory gains over Clarke's theory and the original theory we developed from several viewpoints: ontologically (the explicit distinction between open, semi-open and closed regions is eliminated), definitional (there are fewer defined predicates, and fewer axioms), metatheoretically (there are fewer entries in transitivity tables, and fewer nodes in the the sort hierarchy), and computationally (for comparable theorems, there are fewer formulae in the search space, and fewer nested functions to address in definitions).

The major difference at first sight is ontological parsimony, but we argue that the loss of granularity is not important when modelling physical domains, since physical objects correspond to 'closed' regions, and boundaries can be modelled using 'skins'.

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