Mereology 3

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Overview

- Summary of last weeks class
- Reasoning using countermodels
- Extensionality
- Finite sums and products
- Arbitrary sums and products

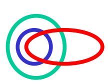
Ground mereology - **M**

- Axioms
  - M1 P xx
  - M2 P xy & P yx ⇒ x = y
  - M3 P xy & P yz ⇒ P xz
- Defined relations:
  - Overlap
  - Underlap
  - Proper part

Assignments due Sep. 10

- **M** |-- (z)(P zx ⇔ P zy) ⇔ x = y
- **M** |-- P xy ⇒ (z)(O zx ⇒ O zy)
- **M** + WSP + SSP |-- PPP
  fill in the gaps in Simon's proof on pg. 29 of 'Parts'
- Give truthtables that show that the given structures are not models Of SSP !

P xy & O xz ⇒ O yz



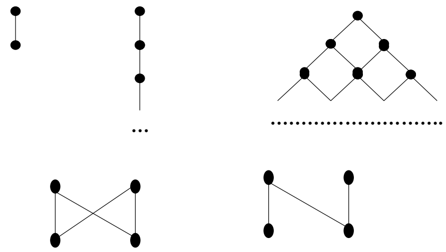
P xy & O xz ⇒ O yz

1.	P xy & O xz	ass
2.	O xz	1 simp
3.	(∃n)(P nx & P nz)	2 D_O
4.	P nx & P nz	3 EI
5.	P nx	4 simp
6.	P xy	1 simp
7.	P nx & P xy	5,6 conj
8.	P nx & P xy ⇒ P ny	M3 UI
9.	P ny	7,8 MP
10.	P nz	4 simp
11.	P ny & P nz	9,10 conj
12.	(∃n)(P ny & P nz)	11 EG
13.	(∃n)(P ny & P nz)	4-12 EI
14.	O yz	13 D_O
15.	P xy & O xz ⇒ O yz	1-14 CP

P xy ⇒ (z)(O xz ⇒ O yz)

1.	P xy	ass
2.	O xz	ass
3.	P xy & O xz	1,2 conj
4.	P xy & O xz ⇒ O yz	Theorem
5.	O yz	3,4 MP
6.	O xz ⇒ O yz	2-5 CP
7.	(z)(O xz ⇒ O yz)	6 UG
8.	P xy ⇒ (z)(O xz ⇒ O yz)	1-7 CP

Ugly models of ground mereology



Extending ground mereology

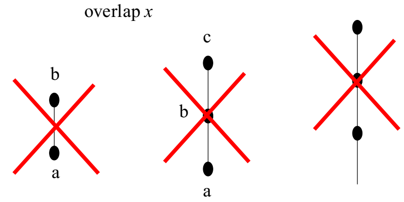
by adding Principles asserting the existence of entities given the existence of other entities

- Whenever an entity has one proper part then it has more than one proper part
- Given two entities then there exists an entity which is the sum of them
- Given a set of entities then there exists an entity that is the sum of the entities in that set
- Products, complements, …

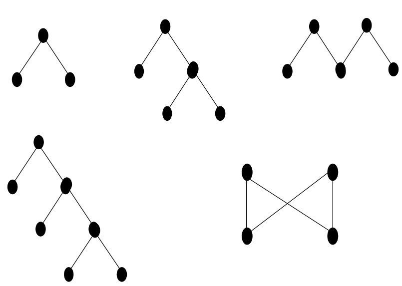
Weak supplementation principle

WSP: $PP\ xy \Rightarrow (\exists z)(PP\ zy \;\&\; \neg O\ zx)$

If *x* is a proper part of *y* then
there is a *z* which is a proper part of *y* and *z* does not overlap *x*

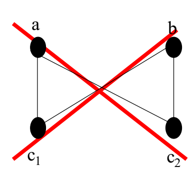


Models of the WSP



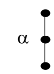
The proper part principle (PPP)

- If
  - *x* has some proper part and
  - Every proper part of *x* is a proper part of *y*
- Then *x* is a part of *y*
- $((\exists z)PP\ zx \;\&\; (\forall z)(PP\ zx \Rightarrow PP\ zy)) \Rightarrow P\ xy$

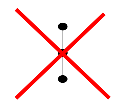


Reasoning using counter models

Reasoning using counter models

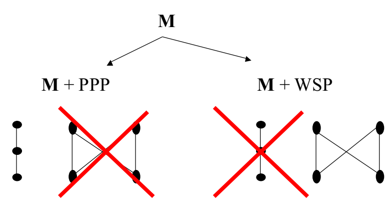


- α is a model of **M**
- α satisfies all axioms in **M**
- Also: α satisfies ALL theorems of **M**

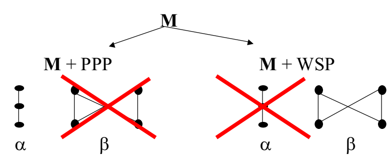


- if α does NOT satisfy Φ
- then Φ cannot be a theorem of **M**
- therefore Φ cannot be proven from **M**

PPP and WSP are independent

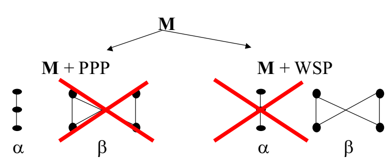


PPP and WSP are independent (2)



- β does NOT satisfy PPP
- β does satisfy **M**+WSP
- thereofore PPP cannot be a theorem of **M**+WSP
- therefore PPP cannot be proven from **M**+WSP

PPP and WSP are independent (3)



- α does NOT satisfy WSP
- α does satisfy **M**+PPP
- thereofore WSP cannot be a theorem of **M**+PPP
- therefore WSP cannot be proven from **M**+PPP

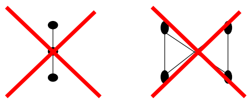
Assignments due Sep. 10

- **M** |-- (z)(P zx ⇔ P zy) ⇔ x = y
- **M** |-- P xy ⇒ (z)(O zx ⇒ O zy)
- **M** + WSP + SSP |-- PPP
  fill in the gaps in Simon's proof on pg. 29 of 'Parts'
- Give truthtables that show that the given structures are not models Of SSP !

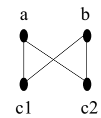
SSP

$\neg P\ xy \Rightarrow (\exists z)(P\ zx \ \& \ \neg O\ zy)$

M + SSP

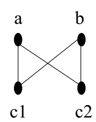


$\neg P\ xy \Rightarrow (\exists z)(P\ zx \ \& \ \neg O\ zy)$



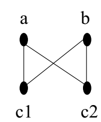
¬P ab	z	⇒	Pza	&	¬O zb
T	a	F	T	F	F

$\neg P\ xy \Rightarrow (\exists z)(P\ zx \ \& \ \neg O\ zy)$



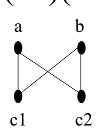
¬P ab	z	⇒	Pza	&	¬O zb
T	a	F	T	F	F
T	b	F	F	F	F

$\neg P\ xy \Rightarrow (\exists z)(P\ zx \ \& \ \neg O\ zy)$



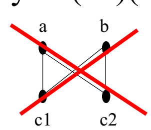
¬P ab	z	⇒	Pza	&	¬O zb
T	a	F	T	F	F
T	b	F	F	F	F
T	c1	F	T	F	F

$\neg P\ xy \Rightarrow (\exists z)(P\ zx \ \& \ \neg O\ zy)$



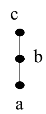
¬P ab	z	⇒	Pza	&	¬O zb
T	a	F	T	F	F
T	b	F	F	F	F
T	c1	F	T	F	F
T	c2	F	T	F	F

$(x)(y)(\neg P\ xy \Rightarrow (\exists z)(P\ zx \ \& \ \neg O\ zy))$



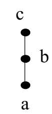
¬P ab	z	⇒	Pza	&	¬O zb
T	a	F	T	F	F
T	b	F	F	F	F
T	c1	F	T	F	F
T	c2	F	T	F	F

¬P xy ⇒(∃z)(P zx & ¬O zy)



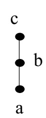
¬P ca	z	⇒	Pzc	&	¬O za
T	a	F	T	F	F

¬P xy ⇒(∃z)(P zx & ¬O zy)



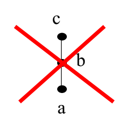
¬P ca	z	⇒	Pzc	&	¬O za
T	a	F	T	F	F
T	b	F	T	F	F

¬P xy ⇒(∃z)(P zx & ¬O zy)



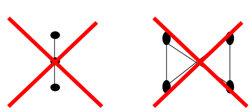
¬P ca	z	⇒	Pzc	&	¬O za
T	a	F	T	F	F
T	b	F	T	F	F
T	c	F	T	F	F

(x)(y)(¬P xy ⇒(∃z)(P zx & ¬O zy))



¬P ca	z	⇒	Pzc	&	¬O za
T	a	F	T	F	F
T	b	F	T	F	F
T	c	F	T	F	F

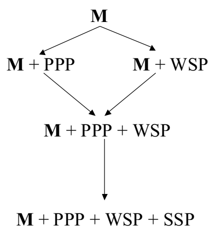
SSP
¬P xy ⇒(∃z)(P zx & ¬O zy)
indeed rules out the models below!



M + PPP + WSP not |-- SSP

- Find a structure that is a model of **M** + PPP + WSP but not of SSP
- All half-open, half closed intervals of the real line: [0,1), [1,2), …,(0,1],(1,2]
- We say last week
  - **M** + PPP + WSP are satisfied
  - SSP is not satisfied
- thereofore SSP cannot be a theorem of **M**+PPP+WSP
- therefore SSP cannot be proven from **M**+PPP+WSP

Hierarchy of theories



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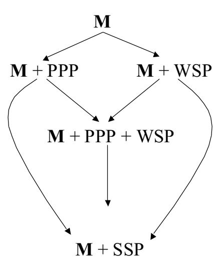
- **M** |-- P xy ⇒ (z)(O zx ⇒ O zy)
- **M** + SSP |-- PPP
  fill in the gaps in Simon's proof on pg. 29 of 'Parts'
- Give truthtables that show that the given structures are not models Of SSP !

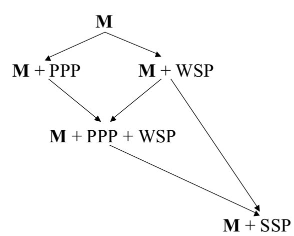
Hierarchy of theories



To prove:

M+SSP |-- PPP
M+SSP |-- WSP
assignment

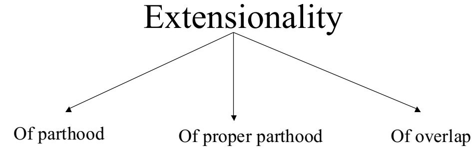
Hierarchy of theories Simons:



Simon's proof of M + SSP |-- PPP

- See your handouts …

Extensionality



Extensionality of parthood

- (z)(P zx ⇔ P zy) ⇔ x = y
- Reflects the view that an object is exhaustively defined by its parts
- Does not commit us to much since parthood includes identity
- Provable using antisymmetry a reflexivity

Assignments due Sep. 10

- **M** |-- (z)(P zx ⇔ P zy) ⇔ x = y
- **M** |-- P xy ⇒ (z)(O zx ⇒ O zy)
- **M** + SSP |-- PPP
  fill in the gaps in Simon's proof on pg. 29 of 'Parts'
- Give truthtables that show that the given structures are not models Of SSP !

x = y ⇒ (z)(P zx ⇔ P zy)

1.	x = y	ass
2.	P zx	ass
3.	P zy	1,2 Id
4.	P zx ⇒ Pzy	2-3 CP
5.	P zy	ass
6.	P zx	1,5 Id
7.	P zy ⇒ P zx	5-6 CP
8.	P zx ⇒ Pzy & P zy ⇒ P zx	4,7 conj
9.	P zx ⇔ P zy	8 Eq
10.	(z)(P zx ⇔ P zy)	9 UG
11.	x = y ⇒ (z)(P zx ⇔ P zy)	1-10 CP

(z)(P zx ⇔ P zy) ⇒ x = y

1.	(z)(P zx ⇔ P zy)	ass
2.	P xx ⇔ P xy	1 UI
3.	(P xx ⇒ P xy) & (P xy ⇒ P xx)	2 Eq
4.	P xx ⇒ P xy	3 simp
5.	(x)P xx	M1
6.	P xx	5 UI
7.	P xy	4,6 MP
8.	P yx ⇔ P yy	1 UI
9.	P yy ⇒ P yx	(8 Eq) simp
10.	P yy	5 UI
11.	P yx	9,10 MP
12.	P xy & P yx	7, 11 conj
13.	P xy & P yx ⇒ x = y	M2 UI
14.	x = y	12,13 MP
15.	(z)(P zx ⇔ P zy) ⇒ x = y	1-14 CP

Extensionality of proper parthood

- (∃z)PP zx & (z)(PP zx ⇔ PP zy) ⇔ x = y
- Reflects the view that an object is exhaustively defined by its constituting parts
- Derivable from PPP
- Problems: perduring entities gain and loose parts all the time and yet remain the same thing

((∃z)PP zx & (∀z)(PP zx ⇔ PP zy)) ⇒ x = y

1.	((∃z)PP zx & (∀z)(PP zx ⇔ PP zy))	ass
2.	(∀z)(PP zx ⇔ PP zy)	1 simp
3.	PP zx ⇔ PP zy	2 UI
4.	PP zx ⇒ PP zy & PP zy ⇒ PP zx	3 Eq
5.	(z)(PP zx ⇒ PP zy)	(4 simp) UG
6.	((∃z)PP zx & (z)(PP zx ⇒ PP zy)	(1 simp),5 conj
7.	P xy	6, PPP MP
8.	(z)(P zy ⇒ PP zx)	(4 simp) UG
9.	(∃z)PP zx	1 simp
10.	PP zx	
11.	PP zx & P xy	10, 7 conj
12.	PP zy	≈11, M3 MP
13.	(∃z) PP zy	12 EG
14.	(∃z) PP zy & (z)(PP zy ⇒ PP zx)	13, 8 conj
15.	P yx	14, PPP MP
16.	P xy & P yx	7,15 conj
17.	x = y	16, M2 MP
18.	((∃z)PP zx & (∀z)(PP zx ⇔ PP zy)) ⇒ x = y	1-17 CP

Extensionality of overlap

- $(z)(O\,zx \Leftrightarrow O\,zy) \Leftrightarrow x = y$
- Reflects the view that two entities are identical if and only if they overlap the same things
- Derivable from SSP
- Problems: perduring entities gain and loose parts – I.e., overlap different objects at different times – and yet remain the same thing

$(z)(O\,zx \Rightarrow O\,zy) \Rightarrow P\,xy$

1.	$\neg P\,xy \Rightarrow (\exists z)(P\,zx \,\&\, \neg O\,zy)$	SSP
2.	$\neg(\exists z)(P\,zx \,\&\, \neg O\,zy) \Rightarrow \neg\neg P\,xy$	1 transp
3.	$(z)\,\neg(P\,zx \,\&\, \neg O\,zy) \Rightarrow P\,xy$	2 DN, QN
4.	$(z)(P\,zx \Rightarrow O\,zy) \Rightarrow P\,xy$	3 Imp
5.	$(z)(O\,zx \Rightarrow O\,zy)$	ass
6.	$O\,zx \Rightarrow O\,zy$	5 UI
7.	$P\,xz \Rightarrow O\,xz$	Theorem
8.	$P\,xz \Rightarrow O\,zy$	7,6 HS
9.	$(z)(P\,xz \Rightarrow O\,zy)$	8 UG
10.	$P\,xy$	9, 4 MP
11.	$(z)(O\,zx \Rightarrow O\,zy) \Rightarrow P\,xy$	5-10 CP

$(\forall z)(O\,zx \Leftrightarrow O\,zy) \Rightarrow x=y$

1.	$(\forall z)(O\,zx \Leftrightarrow O\,zy)$	ass
2.	$O\,zx \Leftrightarrow O\,zy$	1 UI
3.	$O\,zx \Rightarrow O\,zy \,\&\, O\,zy \Rightarrow O\,zx$	2 Eq
4.	$(\forall z)(O\,zx \Rightarrow O\,zy)$	(3 simp)UG
5.	$(z)(O\,zx \Rightarrow O\,zy) \Rightarrow P\,xy$	Theorem UI
6.	$P\,xy$	4,5 MP
7.	$(z)(O\,zy \Rightarrow O\,zx)$	(3 simp)UG
8.	$(z)(O\,zy \Rightarrow O\,zx) \Rightarrow P\,yx$	Theorem UI
9.	$P\,yx$	7,8 MP
10.	$P\,xy \,\&\, P\,yx$	6,9 conj
11.	$x = y$	10, M2 MP
12.	$(\forall z)(O\,zx \Leftrightarrow O\,zy) \Rightarrow x=y$	1- 11 CP

Closure principles:

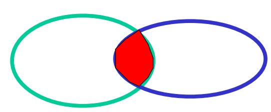
Binary sums, products, differences, and the complement

Extending ground mereology

- Adding Principles asserting the existence of entities given the existence of other entities
  - Whenever an entity has one proper part then it has more than one proper part
  - Given two overlapping entities then there exists an entity which is the product of them and given two entities then there exists an entity which is the sum of them
  - Given a set of entities then there exists an entity that is the sum of the entities in that set

Products in set theory

- Remember set theory: the product of the sets A and B is the set C which contains all the elements which are elements of A **and** elements of B



Important: for any two sets there is a unique set which is the product

Products in Mereology

- There is no counterpart to the empty set in mereology
- Therefore a product only exists if two entities overlap
- If the two entities a and b overlap then the product of a and b is an entity c which is such that for any w if w is a part of c then w is part of a and part of b:
  prod(abc) ≡ (∀w)(P wc ⇔ Pwa & Pwb)

The binary product axiom

- If two entities x and overlap then there exists an entity z which is such that whatever is part of z is also part of x and y and vice versa
- A_{prod} O xy ⇒ (∀w)(P wc ⇔ Pwa & Pwb)
- A_{prod} O xy ⇒(∃z) prod(xyz)
  - This ensures that products for overlappers always exist
  - From extensionality of parthood it follows that that products are unique:
    prod (xyz_1) & prod (xyz_2) ⇒ $z_1=z_2$

prod (xyz_1) & prod (xyz_2) ⇒ $z_1=z_2$

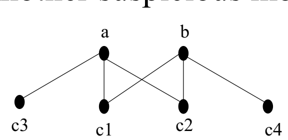
We use: (z)(P zx ⇔ Pzy) ⇔ x=y

1.	prod (xyz_1) & prod (xyz_2)	ass
2.	prod (xyz_1)	1 simp
3.	P wz_1 ⇔ P wx & P wy	(2 D_{prod}) UI
4.	P wz_1 ⇒ P wx & P wy	(3 Eq) simp
5.	P wz_1	ass
6.	P wx & P wy	4,5 MP
7.	Prod (xyz_2)	1 simp
8.	P wz_2 ⇔ P wx & P wy	(7 D_{prod}) UI
9.	P wx & P wy ⇒ P wz_2	(8 Eq) simp
10.	P wz_2	6,9 MP
11.	P wz_1 ⇒ P wz_2	5-10 CP
12.	P wz_2	ass
13.	… like 5-9 above	
14.	P wz_1	
15.	P wz_2 ⇒ P wz_1	12-14 CP
16.	P wz_1 ⇔ P wz_2	(11,15 conj) Eq
17.	(w)(P wz_1 ⇔ P wz_2)	16 UG
18.	$z_1=z_2$	17, 0 MP
19.	prod (xyz_1) & prod (xyz_2) ⇒ $z_1=z_2$	1-19 CP

Binary products

- The binary product axiom ensures that if two entities overlap then they have a product
- From extensionality of parthood it follows that products are unique when they exist
- Therefore prod (xyz) is a partial function and we can write z = x * y

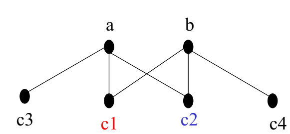
Another suspicious model:



Satisfies SSP:

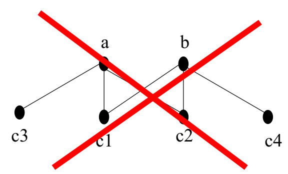
- a and b are distinct since they have some parts not on common
- P c_3a & ¬P c_4a
- P c_4b & ¬P c_3b

Another suspicious model (2)



- Problem: Two entities that overlap should have a unique product!
- **But**: c1 and c2 are equally good candidates for the product of a and b:
  - (∀w)P wc1 ⇔ Pwa & Pwb, i.e., prod (abc1)
  - (∀w)P wc2 ⇔ Pwa & Pwb, i.e., prod (abc2)

The binary product axiom



- Uniqueness of products rules out this model

The ι operator

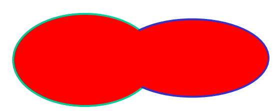
- $a*b \equiv (\iota z)(\forall w)(P\ wz \Leftrightarrow Pwa\ \&\ Pwb)$
- (ιz) means that there exists exactly one z
- Russell operator
- $(\iota x)(\Phi x)$ is considered as an entity
  - $z = (\iota x)(\Phi x)$
  - z is identical to the unique x for which Φ holds
- $\Psi(\iota x)(\Phi x) \Leftrightarrow (\exists x)\ \{\Phi x\ \&\ (\forall y)(\Phi y \Rightarrow y{=}x)\ \&\ \Psi x\}$
  - $z = (\iota x)(\Phi x)$ is equivalent to $(\exists x)\ \{\Phi x\ \&\ (\forall y)(\Phi\ y \Rightarrow y{=}x)\ \&\ x{=}z\}$

Stronger axioms

- Use the definition
  $a*b \equiv (\iota z)(\forall w)(P\ wz \Leftrightarrow Pwa\ \&\ Pwb)$
- Write the product axiom as
  - A_* $O\ xy \Rightarrow (\exists z)(z = x*y)$
- Here the uniqueness of products follows directly from A_*
- SSP becomes derivable (see Simon's proof on pg. 31 of Parts)

Sum in set theory

- Remember set theory: the sum of the sets A and B is the set C which contains all the elements which are either elements of A **or** elements of B



Important: for any two sets there is a unique set which is the sum

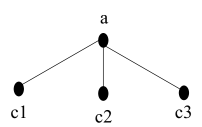
Sums in Mereology

- Do not always exist since there does not need to exist a universe which is the sum of all entities
- Therefore a sum only exists if two entities underlap
- If the two entities a and b underlap then the sum of a and b is an entity c which is such that for any w: if w overlaps c then w overlaps a or w overlaps b and vice versa:
  $sum(abc) \equiv (\forall w)(O\ wc \Leftrightarrow O\ wa \text{ or } O\ wb)$

The binary sum axiom

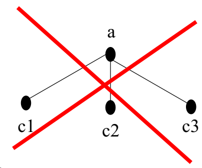
- If two entities x and underlap then there exists an entity z which is such that whatever is overlaps z is also overlaps x or y and vice versa
- A_{sum} $U\ xy \Rightarrow (\forall w)(O\ wc \Leftrightarrow O\ wa \text{ or } O\ wb)$
- A_{sum} $U\ xy \Rightarrow (\exists z)\ sum(xyz)$
  - This ensures that sums for underlappers always exist
  - From extensionality of overlap it follows that that sums are unique:
    $sum\ (xyz_1)\ \&\ sum\ (xyz_2) \Rightarrow z_1{=}z_2$

Again a suspicious (?) model:



Satisfies M, SSP, A_{prod}

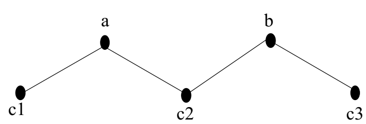
Again a suspicious (?) model:



Ruled out by A_{sum}:

- c_1 and c_2 underlap but NOT sum(c_1c_2a):
- Not everything that overlaps a also overlaps c_1 or c_2 : c_3

No universe !
(no entity which all entities as parts)



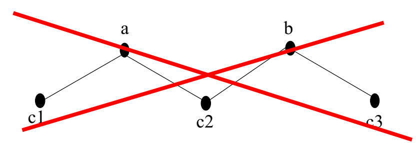
Satisfies M, SSP, A_{prod}, A_{sum}

U $c_1c_2 \Rightarrow$ sum (c_1c_2a)

U $c_2c_3 \Rightarrow$ sum (c_2c_3b)

O ab $\Rightarrow$ prod (abc$_2$)

The universe exists !



There exists an entity which has all entities of the domain as its parts:

A_U $(\exists y)(\forall x)$ Pxy

Consequences of $(\exists y)(\forall x)$ Pxy

- Any two entities in the domain underlap since everything is part of the universe
- The premise in U xy $\Rightarrow(\exists z)$ sum(xyz) can be dropped
- In the presence of extensionality we can prove that the universe is unique
- The universe then can be defined as U $\equiv (\iota y)(\forall x)$ Pxy

Stronger axioms

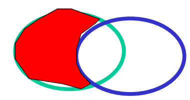
- Use the definition
  a+b $\equiv(\iota z)(\forall w)$(O wz $\Leftrightarrow$ O wa or O wb)
- Write the sum axiom as
  - A_+ U xy $\Rightarrow (\exists z)(z = x+y)$
- Here the uniqueness of sums follows directly from A_+

Strange entities

- Assume the universe exists then we have
  $(\forall x)(\forall y)(\exists z)(z = x+y)$
- Example sums
  - The sum me and George W.
  - The sum of my nose and the Eiffel Tower
  - The sum of my pen and the number 1

Set theoretic difference

- Set-theoretical difference of A and B: is the set C which has all elements of A which are not elements of B



Important: for any two sets there is a unique set which is the sum

Mereological difference

- z is the difference of a and b iff everything which is part of z is also part of a but does not overlap b and vice versa
- $a - b \equiv (\iota z)(\forall w)(P\ wz \Leftrightarrow P\ wa \;\&\; \neg O\ wb)$

Remainder principle (RP)

- If x is not a part of y then there exists a set which is the difference of x and y
- $\neg P\ xy \Rightarrow (\exists z)(z = x\text{-}y)$
- RP implies SSP
- SSP implies RP ???

RP $\Rightarrow$ SSP

1.	$\neg P\ xy$	ass
2.	$(\exists z)(z=x\text{-}y)$	1, RP MP
3.	$(\exists z)(w)(P\ wz \Leftrightarrow (P\ wx \;\&\; \neg Owy))$	2 D_
4.	$(w)(P\ wz \Leftrightarrow (P\ wx \;\&\; \neg Owy))$	
5.	$P\ zz \Rightarrow (P\ zx \;\&\; \neg Ozy)$	(4 UI) EQ
6.	$P\ zx \;\&\; \neg Ozy$	M1, 5 MP
7.	$(\exists z)(P\ zx \;\&\; \neg Ozy)$	6 EG
8.	$(\exists z)(P\ zx \;\&\; \neg Ozy)$	3-7 EI
9.	$\neg P\ xy \Rightarrow (\exists z)(P\ zx \;\&\; \neg Ozy)$	1-9 CP

Mereological complement

- The complement of x is the entity z such that all parts of z are disjoint from (do not overlap) x and everything that is disjoint from x is a part of z
- $\sim x \equiv (\iota z)(\forall w)(P\ wz \Leftrightarrow \neg O\ wx)$
- Complementation principle
  - $(\exists z)(\neg P\ zx) \Rightarrow (\exists z)(z = \sim x)$
  - Independent from PPP, WSP, SSP, RP

Extending ground mereology

- Adding Principles asserting the existence of entities given the existence of other entities
  - Whenever an entity has one proper part then it has more than one proper part
  - Given two overlapping entities then there exists an entity which is the product of them and given two entities then there exists an entity which is the sum of them
  - Given a set of entities then there exists an entity that is the sum of the entities in that set

Unrestricted fusions

- Allow sums for arbitrary non-empty sets of entities
- Problem: we cannot quantify over sets of entities in a first order theory
- Avoid explicit reference to sets by using *axiom schemata* that involve that involve only predicates of open formulas

Axiom schemata

- $(\exists x)\varphi(x) \Rightarrow (\exists z)(w)(O\ wz \Leftrightarrow (\exists x)(\varphi(x)\ \&\ O\ wx))$
  - Abbreviation: $(\exists x)\varphi(x) \Rightarrow (\exists z)$ z Sum x φ
  - z Sum x φ means that z is the sum of all x that satisfy φ
- $\varphi(x)$ stands for any first order formula in which the variable x occurs free (not bound by a quantifier)
- Axiom schemata means that for any formula φ the is an axiom ensuring the existence of the sum of the entities satisfying φ.

Axiom schemata (2)

- Examples for instantiations of $(\exists x)\varphi(x) \Rightarrow (\exists z)$ Sum x φ
  - $(\exists x)Pxx \Rightarrow (\exists z)$ Sum x Pxx
    the sum of all entities that are parts of themselfes
  - $(\exists x)P\ xy \Rightarrow (\exists z)$ Sum x P xy
    the sum of all entities x that are part of y
  - $(\exists x)P\ yx \Rightarrow (\exists z)$ Sum x P yx
    the sum of all entities x of which y is part of
  - …

The summation axiom

- z Sum x φ means:
  - z is the sum of all x that satisfy φ
- z Sum x $\varphi \equiv$
  - $(w)(O\ wz \Leftrightarrow (\exists x)(\varphi(x)\ \&\ O\ yw))$
  - Anything overlaps z iff there exists an entity x that satisfies φ and that overlaps w
- The summation axiom
  - $(\exists x)\varphi(x) \Rightarrow (\exists z)$ z Sum x φ
  - Whatever φ there is if there is one thing that satisfies φ then there exists the sum of all φ-ers

Uniqueness of summation

- In the presence of extensionality of overlap then sums are unique
- z_1 Sum x φ & z_2 Sum x $\varphi \Rightarrow z_1 = z_2$
- Prove this at home

Stronger axioms

- Use the definition
  - $z \text{ Sum } x\ \varphi \equiv (\iota z)(w)(O\ wz \Leftrightarrow (\exists x)(\varphi(x)\ \&\ O\ yw))$
- Write the sum axiom as
  - A_{Sum} $(\exists z)(z = \text{Sum } x\ \varphi)$
- Here the uniqueness of sums follows directly from A_{Sum}

Strength of the summation axiom

- $x+y \equiv \text{Sum } z\ (P\ zx \text{ or } P\ zy)$
- $x * y \equiv \text{Sum } z\ (P\ zx\ \&\ P\ zy)$
- $x - y \equiv \text{Sum } z\ (P\ zx\ \&\ \neg O\ zy)$
- $\sim x \equiv \text{Sum } z\ (\neg O\ zx)$
- $U = \text{Sum } z\ (P\ zz)$

More strange entities

- The sum of me and the real numbers
- The sum of all humans and all tables
- …

Summary and assignments

Ground mereology - **M**

- Axioms
  - M1 P xx
  - M2 P xy & P yx ⇒ x = y
  - M3 P xy & P yz ⇒ P xz
- Defined relations:
  - Overlap
  - Underlap
  - Proper part

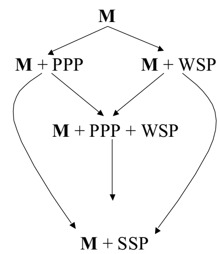
Extending ground mereology

- Adding Principles asserting the existence of entities given the existence of other entities
  - Whenever an entity has one proper part then it has more than one proper part
  - Given two overlapping entities then there exists an entity which is the product of them and given two entities then there exists an entity which is the sum of them
  - Given a set of entities then there exists an entity that is the sum of the entities in that set

Whenever an entity has one proper part then it has more than one proper part

- WSP
  - $PP\,xy \Rightarrow (\exists z)(P\,zy \,\&\, \neg O\,zx)$
- PPP
  - $((\exists z)PP\,zx \,\&\, (\forall z)(PP\,zx \Rightarrow PP\,zy)) \Rightarrow P\,xy$
- SSP
  - $\neg P\,xy \Rightarrow (\exists z)(P\,zx \,\&\, \neg O\,zy)$
- RP
  - $\neg P\,xy \Rightarrow (\exists z)(z = x\text{-}y)$

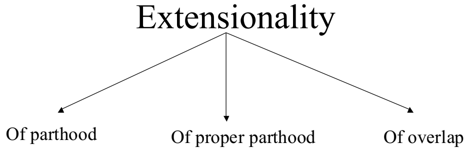
Hierarchy of theories



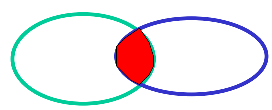
To prove:

M+SSP |-- PPP
M+SSP |-- WSP
assignment

Extensionality



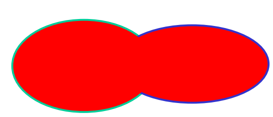
Binary Products



Products in Mereology

- There is no counterpart to the empty set in mereology
- Therefore a product only exists if two entities overlap
- If the two entities a and b overlap then the product of a and b is an entity c which is such that for any w if w is a part of c then w is part of a and part of b:
  prod(abc) ≡ (∀w)(P wc ⇔ Pwa & Pwb)

Binary Sums



Sums in Mereology

- Do not always exist since there does not need to exist a universe which is the sum of all entities
- Therefore a product only exists if two entities underlap
- If the two entities a and b underlap then the sum of a and b is an entity c which is such that for any w: if w overlaps c then w overlaps a or w overlaps b and vice versa: sum(abc) ≡ (∀w)(O wc ⇔ O wa or O wb)

The unrestricted summation axiom

- z Sum x φ means:
  - z is the sum of all x that satisfy φ
- z Sum x φ ≡
  - (w)(O wz ⇔(∃x)(φ(x) & O yw))
  - Anything overlaps z iff there exists an entity x that satisfies φ and that overlaps w
- The summation axiom
  - (∃x)φ(x) ⇒ (∃z) z Sum x φ
  - Whatever φ there is if there is one thing that satisfies φ then there exists the sum of all φ-ers

Strength of the summation axiom

- x+y ≡ Sum z (P zx or P zy)
- x * y ≡ Sum z (P zx & P zy)
- x – y ≡ Sum z (P zx & ¬O zy)
- ~x ≡ Sum z (¬O zx)
- U = Sum z (P zz)

Assignments due Wd. 17

- **M**+SSP |-- WSP
- Prove the uniqueness of binary sums (assuming extensionality of O):
  sum (abz$_1$) & sum (abz$_2$) ⇒ $z_1 = z_2$
- Prove the uniqueness of arbitrary sums (assuming extensionality of O):
  z_1 Sum x φ & z_2 Sum x φ ⇒ $z_1 = z_2$