Mereology 4

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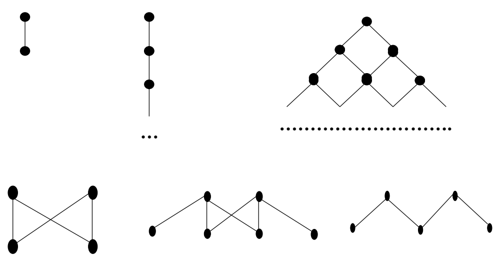
Ground mereology - **M**

- Axioms
  - M1 P xx
  - M2 P xy & P yx ⇒ x = y
  - M3 P xy & P yz ⇒ P xz
- Defined relations:
  - Overlap
  - Underlap
  - Proper part

Ugly models of ground mereology



Extending ground mereology

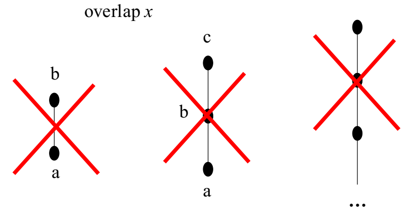
- Adding Principles asserting the existence of entities given the existence of other entities
  - Whenever an entity has one proper part then it has more than one proper part
  - Given two overlapping entities then there exists an entity which is the product of them and given two entities then there exists an entity which is the sum of them
  - Given a set of entities then there exists an entity that is the sum of the entities in that set

Whenever an entity has one proper part then it has more than one proper part

Weak supplementation principle

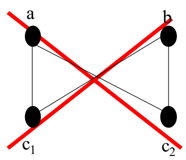
WSP: PP xy ⇒ (∃z)(PP zy & ¬O zx)

If *x* is a proper part of *y* then
there is a *z* which is a proper part of *y* and *z* does not overlap *x*

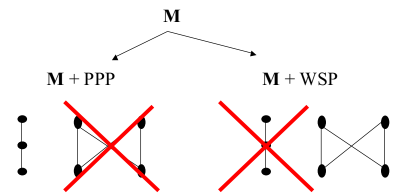


The proper part principle (PPP)

- If
  - *x* has some proper part and
  - every proper part of *x* is a proper part of *y*
- Then *x* is a part of *y*
- $((\exists z)PP\ zx\ \&\ (\forall z)(PP\ zx \Rightarrow PP\ zy)) \Rightarrow P\ xy$

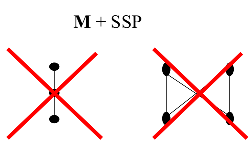


PPP and WSP are independent



SSP

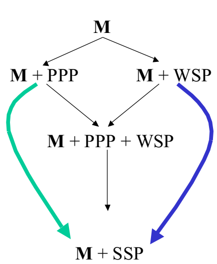
$$\neg P\ xy \Rightarrow (\exists z)(P\ zx\ \&\ \neg O\ zy)$$



M + PPP + WSP not |-- SSP

- Find a structure that is a model of **M** + PPP + WSP but not of SSP
- All half-open, half closed intervals of the real line: [0,1), [1,2), …,(0,1],(1,2]
- We saw two weeks ago
  - **M** + PPP + WSP are satisfied
  - SSP is not satisfied
- thereofore SSP cannot be a theorem of **M**+PPP+WSP
- therefore SSP cannot be proven from **M**+PPP+WSP

Hierarchy of theories



We proved:

M+SSP |-- PPP

M+SSP |-- WSP

This weeks assignment

M+SSP |-- $PP\ xy \Rightarrow (\exists z)(PP\ zy\ \&\ \neg O\ zx)$

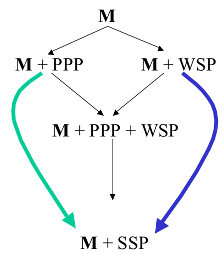
assumptions:

0	$\neg P\ yx \Rightarrow (\exists z)(P\ zy\ \&\ \neg O\ zx)$	SSP
0a	$P\ xy \Rightarrow O\ xy$	Th
0b	$Oxy \Rightarrow O\ yx$	Th

M+SSP |-- PP xy ⇒ (∃z)(PP zy & ¬O zx)

1.	PP xy	ass
2.	P xy & ¬ P yx	1 D_{PP}
3.	¬ P yx	2 simp
4.	(∃z)(P zy & ¬O zx)	3,0 MP
5.	P zy & ¬O zx	
6.	P zy	5 simp
7.	P yz	ass
8.	P zy & P yz	6,7 conj
9.	y=z	8, M2 MP
10.	¬O zx	5 simp
11.	¬O yx	9,10 Id
12.	P xy	2 simp
13.	O xy	12, 0a MP
14.	O yx	13, 0b MP
15.	O yx & ¬O yx	14,11 conj
16.	¬ P yz	7-15 IP
17.	P zy & ¬ P yz	6, 16 conj
18.	PP zy	17 D_{PP}
19.	PP zy & ¬O zx	18, 10 conj
20.	(∃z)(PP zy & ¬O zx)	19 EG
21.	PP xy ⇒ (∃z)(PP zy & ¬O zx)	1-20 CP

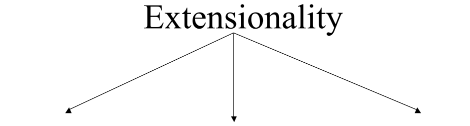
Hierarchy of theories



We proved:

M+SSP |-- PPP
M+SSP |-- WSP
This weeks assignment

Extensionality



Extensionality

Of proper parthood

- (∃z)PP zx & (z)(PP zx ⇔ PP zy) ⇔ x = y
- an object is exhaustively defined by its constituting parts
- Follows immediately from PPP

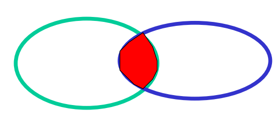
Of overlap

- (z)(O zx ⇔ O zy) ⇔ x = y
- two entities are identical if and only if they overlap the same things
- Follows immediately from SSP

Extending ground mereology

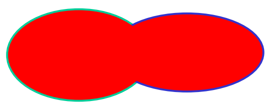
- Adding Principles asserting the existence of entities given the existence of other entities
  - Given two overlapping entities then there exists an entity which is the product of them and
  - given two entities then there exists an entity which is the sum of them
- Closure principles

Binary products



- If the two entities a and b overlap then the product of a and b is an entity c which is such that for any w if w is a part of c then w is part of a and part of b:
  prod(abc) ≡ (∀w)(P wc ⇔ Pwa & Pwb)

Binary sums



- If the two entities a and b underlap then the sum of a and b is an entity c which is such that for any w: if w overlaps c then w overlaps a or w overlaps b and vice versa: sum(abc) $\equiv$ ($\forall$w)(O wc $\Leftrightarrow$ O wa or O wb)

The binary product axiom

- If two entities x and overlap then there exists an entity z which is such that whatever is part of z is also part of x and y and vice versa
- A_{prod} O xy $\Rightarrow$($\exists$z) prod(xyz)
- A_{prod} ensures that products for overlappers always exist
- From extensionality of parthood it follows that that products are unique:
  prod (xyz_1) & prod (xyz_2) $\Rightarrow$ z_1=z_2
- Product is a partial function: x*y=z

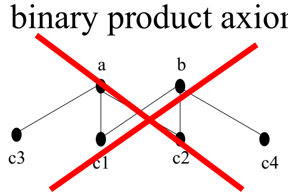
The binary sum axiom

- If two entities x and underlap then there exists an entity z which is such that whatever is overlaps z is also overlaps x or y and vice versa
- A_{sum} U xy $\Rightarrow$($\exists$z) sum(xyz)
- A_{sum} ensures that sums for underlappers always exist
- From extensionality of overlap it follows that that sums are unique:
  sum (xyz_1) & sum (xyz_2) $\Rightarrow$ z_1=z_2
- Sum is a (partial) function: x+y=z

sum (xyz_1) & sum (xyz_2) $\Rightarrow$ z_1=z_2
We use: (z)(O zx $\Leftrightarrow$ Ozy) $\Leftrightarrow$ x=y

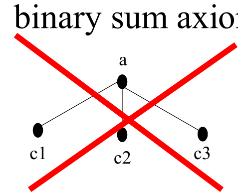
1.	sum (xyz_1) & sum (xyz_2)	ass
2.	sum (xyz_1)	1 simp
3.	O wz_1 $\Leftrightarrow$ O wx or O wy	(2 D_{sum}) UI
4.	O wz_1 $\Rightarrow$ O wx or O wy	(3 Eq) simp
5.	O wz_1	ass
6.	O wx or O wy	4,5 MP
7.	sim (xyz_2)	1 simp
8.	O wz_2 $\Leftrightarrow$ O wx or O wy	(7 D_{sum}) UI
9.	O wx or O wy $\Rightarrow$ O wz_2	(8 Eq) simp
10.	O wz_2	6,9 MP
11.	O wz_1 $\Rightarrow$ O wz_2	5-10 CP
12.	O wz_2	ass
13.	… like 5-9 above	
14.	O wz_1	
15.	O wz_2 $\Rightarrow$ O wz_1	12-14 CP
16.	O wz_1 $\Leftrightarrow$ O wz_2	(11,15 conj) Eq
17.	(w)(O wz_1 $\Leftrightarrow$ O wz_2)	16 UG
18.	z_1=z_2	17, 0 MP
19.	sum (xyz_1) & sum (xyz_2) $\Rightarrow$ z_1=z_2	1-19 CP

Ruled out by the binary product axiom



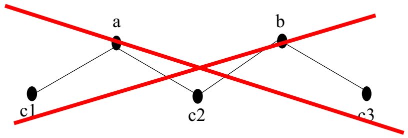
- Uniqueness of products rules out this model

Ruled out by the binary sum axiom



Satisfies M, SSP, A_{prod}

The universe exists !



There exists an entity which has all entities of the domain as its parts:

A_U $(\exists y)(\forall x)\, Pxy$

Consequences of $(\exists y)(\forall x)\, Pxy$

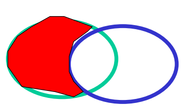
- Any two entities in the domain underlap since everything is part of the universe
- The premise in $U\, xy \Rightarrow (\exists z)\, sum(xyz)$ can be dropped
- In the presence of extensionality we can prove that the universe is unique

Strange entities

- Assume the universe exists then we have
  $(\forall x)(\forall y)(\exists z)(z = x+y)$
- Example sums
  - The sum me and George W.
  - The sum of my nose and the Eiffel Tower
  - The sum of my pen and the number 1

Here starts the new stuff!!

Mereological difference



- z is the difference of a and b iff everything which is part of z is also part of a but does not overlap b and vice versa
- $minus(abz) \equiv (\forall w)(P\, wz \Leftrightarrow P\, wa \ \&\ \neg O\, wb)$
- Ext. of parthood – difference is unique: a-b=z

Remainder principle (RP)

- If x is not a part of y then there exists a z which is the difference of x and y
- $\neg P\, xy \Rightarrow (\exists z)(z = x\text{-}y)$
- RP implies SSP

RP $\Rightarrow$ SSP

1.	$\neg$P xy	ass
2.	$(\exists z)(z=x-y)$	1, RP MP
3.	$(\exists z)(w)(P\ wz \Leftrightarrow (P\ wx\ \&\ \neg Owy))$	2 D_-
4.	$(w)(P\ wz \Leftrightarrow (P\ wx\ \&\ \neg Owy))$	
5.	$P\ zz \Rightarrow (P\ zx\ \&\ \neg Ozy)$	(4 UI) EQ
6.	$P\ zx\ \&\ \neg Ozy$	M1, 5 MP
7.	$(\exists z)(P\ zx\ \&\ \neg Ozy)$	6 EG
8.	$(\exists z)(P\ zx\ \&\ \neg Ozy)$	3-7 EI
9.	$\neg P\ xy \Rightarrow (\exists z)(P\ zx\ \&\ \neg Ozy)$	1-9 CP

SSP implies RP ???

Varzi, A (2003), Mereology, pg. 15

The corresponding closure principles can therefore be stated thus:

(P.8) $\neg Pyx \rightarrow \exists z(z = x-y)$ Remainder

(P.9) $\exists z \neg Pzx \rightarrow \exists z(z = \sim x)$ Complementation

SSP

The first of these is equivalent to (P.5), but the second is independent of any of the principles considered so far. In many versions, a closure theory also involves a postulate to the effect that the domain has an upper bound—that is, there is something (the "universal individual") of which everything is part:

Assignment: prove that SSP implies RP or show that this is impossible

The ι operator

- $a*b \equiv (\iota z)(\forall w)(P\ wz \Leftrightarrow Pwa\ \&\ Pwb)$
- (ιz) means that there exists exactly one z
- Russell operator
- $(\iota x)(\Phi x)$ is considered as an entity
  - $z = (\iota x)(\Phi x)$
  - z is identical to the unique x for which Φ holds
- $\Psi(\iota x)(\Phi x) \Leftrightarrow (\exists x)\ \{\Phi x\ \&\ (\forall y)(\Phi y \Rightarrow y=x)\ \&\ \Psi x\}$
  - $z = (\iota x)(\Phi x)$ is equivalent to $(\exists x)\ \{\Phi x\ \&\ (\forall y)(\Phi\ y \Rightarrow y=x)\ \&\ x=z\}$

Stronger axioms

- Use the definitions
  - $a*b \equiv (\iota z)(\forall w)(P\ wz \Leftrightarrow Pwa\ \&\ Pwb)$
  - $a+b \equiv (\iota z)(\forall w)(O\ wz \Leftrightarrow O\ wa \text{ or } O\ wb)$
- Write the axioms as
  - A_* $\quad O\ xy \Rightarrow (\exists z)(z = x*y)$
  - A_+ $\quad U\ xy \Rightarrow (\exists z)(z = x+y)$

Mereological complement

- The complement of x is the entity z such that all parts of z are disjoint from (do not overlap) x and everything that is disjoint from x is a part of z
- $\sim x \equiv (\iota z)(\forall w)(P\ wz \Leftrightarrow \neg O\ wx)$
- Complementation principle
  - $(\exists z)(\neg P\ zx) \Rightarrow (\exists z)(z = \sim x)$
  - Independent from PPP, WSP, SSP, RP

Extending ground mereology

- Adding Principles asserting the existence of entities given the existence of other entities
  - Whenever an entity has one proper part then it has more than one proper part
  - Given two overlapping entities then there exists an entity which is the product of them and given two entities then there exists an entity which is the sum of them
  - Given a set of entities then there exists an entity that is the sum of the entities in that set

Unrestricted fusions

- Allow sums for arbitrary non-empty sets of entities
- Problem: we cannot quantify over sets of entities in a first order theory
- Avoid explicit reference to sets by using *axiom schemata* that involve that involve only predicates of open formulas

Axiom schemata

- $(\exists x)\varphi(x) \Rightarrow (\exists z)(w)(O\ wz \Leftrightarrow (\exists x)(\varphi(x)\ \&\ O\ wx))$
  - Abbreviation: $(\exists x)\varphi(x) \Rightarrow (\exists z)$ z Sum x φ
  - z Sum x φ means that z is the sum of all x that satisfy φ
- $\varphi(x)$ stands for any first order formula in which the variable x occurs free (not bound by a quantifier)
- Axiom schemata means that for any formula φ there is an axiom ensuring the existence of the sum of the entities satisfying φ.

Axiom schemata (2)

- Examples for instantiations of $(\exists x)\varphi(x) \Rightarrow (\exists z)$ z Sum x φ
  - $(\exists x)Pxx \Rightarrow (\exists z)$ z Sum x Pxx
    the sum of all entities that are parts of themselfes
  - $(\exists x)P\ xy \Rightarrow (\exists z)$ z Sum x P xy
    the sum of all entities x that are part of y
  - $(\exists x)P\ yx \Rightarrow (\exists z)$ z Sum x P yx
    the sum of all entities x of which y is part of
  - …

The summation axiom

- z Sum x φ means:
  - z is the sum of all x that satisfy φ
- z Sum x $\varphi \equiv$
  - $(w)(O\ wz \Leftrightarrow (\exists x)(\varphi(x)\ \&\ O\ xw))$
  - Anything overlaps z iff there exists an entity x that satisfies φ and that overlaps w
- The summation axiom
  - $(\exists x)\varphi(x) \Rightarrow (\exists z)$ z Sum x φ
  - Whatever φ there is if there is one thing that satisfies φ then there exists the sum of all φ-ers

Uniqueness of summation

- In the presence of extensionality of overlap then sums are unique
- z_1 Sum x φ & z_2 Sum x $\varphi \Rightarrow z_1 = z_2$
- Prove this at home

Stronger axioms

- Use the definition
  - z Sum x $\varphi \equiv (\iota z)(w)(O\ wz \Leftrightarrow (\exists x)(\varphi(x)\ \&\ O\ yw))$
- Write the sum axiom as
  - A_{Sum} $\quad (\exists z)(z = \text{Sum x } \varphi)$
- Here the uniqueness of sums follows directly from A_{Sum}

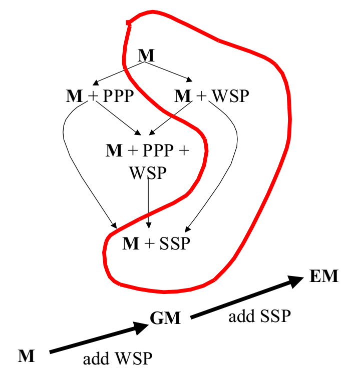
Strength of the summation axiom

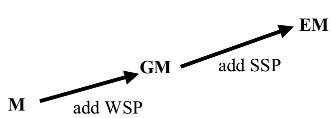
- $x+y \equiv \text{Sum } z\ (P\ zx \text{ or } P\ zy)$
- $x * y \equiv \text{Sum } z\ (P\ zx\ \&\ P\ zy)$
- $x - y \equiv \text{Sum } z\ (P\ zx\ \&\ \neg O\ zy)$
- $\sim x \equiv \text{Sum } z\ (\neg O\ zx)$
- $U = \text{Sum } z\ (P\ zz)$

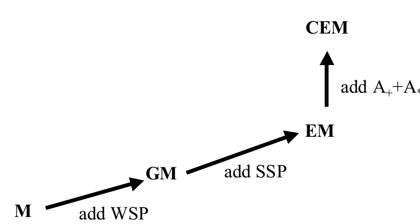
More strange entities

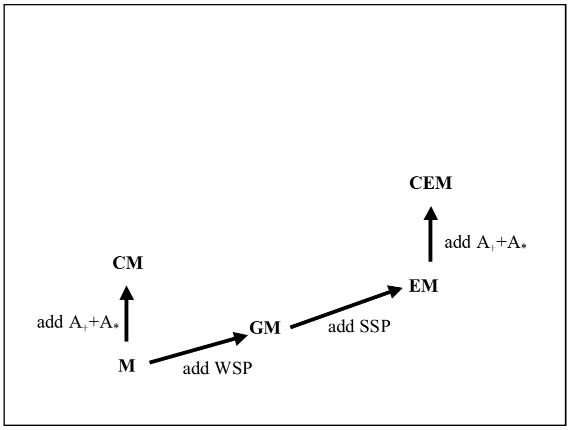
- The sum of me and the real numbers
- The sum of all humans and all tables
- …

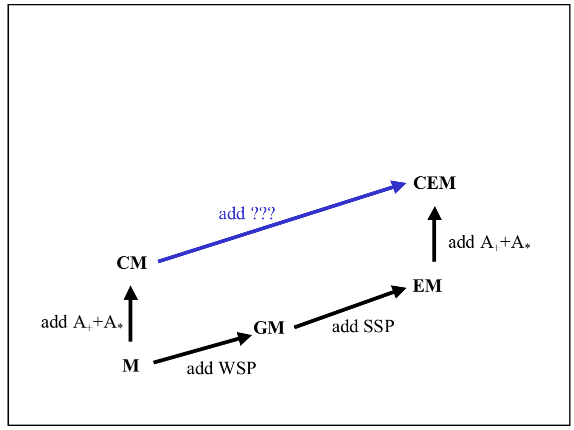
Hierarchy of theories

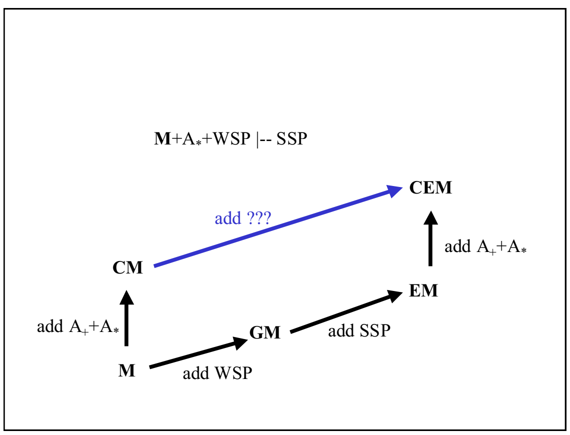







CEM
add $A_+ + A_*$
CM
EM
add $A_+ + A_*$
GM
add SSP
M
add WSP


CEM
add ???
add $A_+ + A_*$
CM
EM
add $A_+ + A_*$
GM
add SSP
M
add WSP


M+A_*+WSP |-- SSP
CEM
add ???
add $A_+ + A_*$
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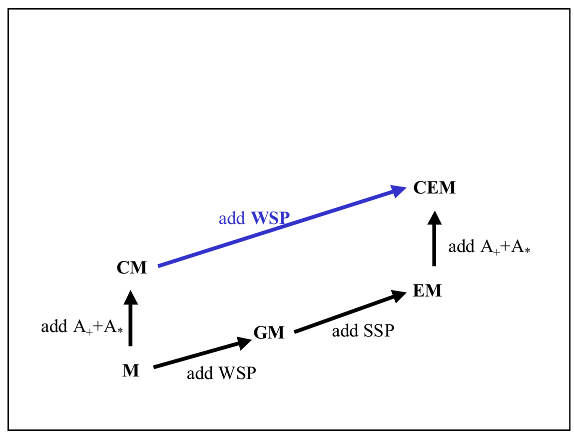
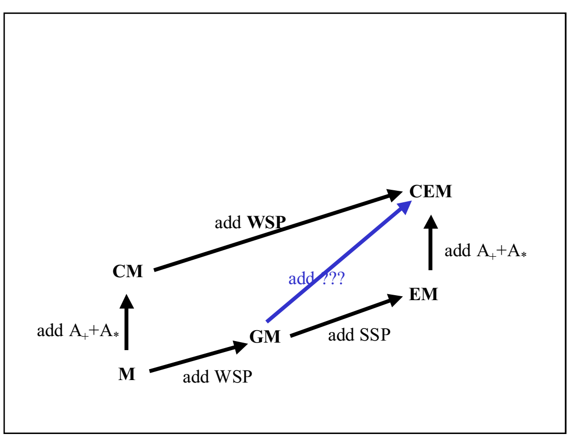

CEM
add **WSP**
add $A_+ + A_*$
CM
EM
add $A_+ + A_*$
GM
add SSP
M
add WSP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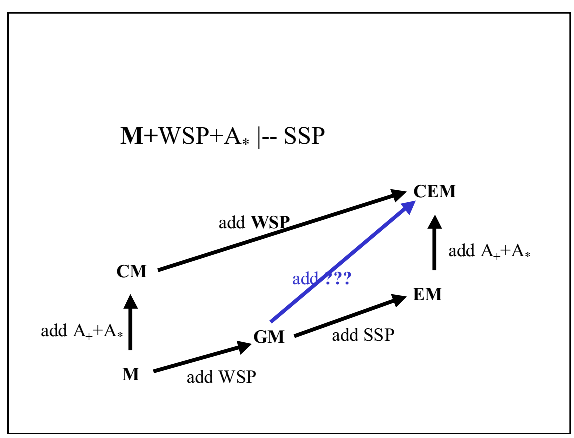
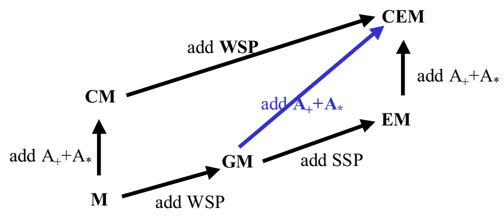

M+WSP+A_* |-- SSP
CEM
add **WSP**
add $A_+ + A_*$
CM
add **???**
EM
add $A_+ + A_*$
GM
add SSP
M
add WSP

M+WSP+A* |-- SSP



M+WSP+A* |-- SSP (See Pontow (2003))

Lemmata

1. O xy ⇒ P x(x*y)
2. ¬ P xy ⇒ (¬ O xy ⇒ (∃z)(P zx & ¬ O zy))
3. ¬ P xy ⇒ (O xy ⇒ (∃z)(P zx & ¬ O zy))
4. (P xy & O yz) ⇒ (O xy ⇒ O x(y*z))
5. PP xy or x = y ⇒ P xy

Theorem

M+WSP+A* |-- ¬ P xy ⇒ (∃z)(P zx & ¬ O zy)

M+WSP+A* |-- SSP

Lemmata

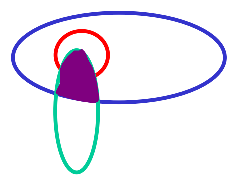
1.	O xy ⇒ P (x*y)x	now
2.	¬ P xy ⇒ (¬ O xy ⇒ (∃z)(P zx & ¬ O zy))	at home
3.	¬ P xy ⇒ (O xy ⇒ (∃z)(P zx & ¬ O zy))	now
4.	P zx ⇒ (O zy ⇒ O z(x*y))	at home
5.	PP xy or x = y ⇒ P xy	done
6.	P xy & ¬x=y ⇒ PP xy	done

Theorem

M+WSP+A* |-- ¬ P xy ⇒ (∃z)(P zx & ¬ O zy)

at home

P zx ⇒ (O zy ⇒ O z(x*y))

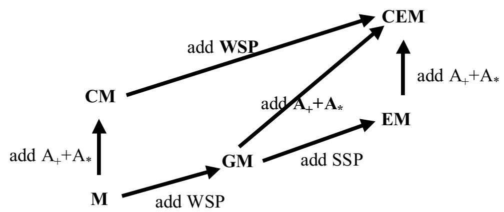


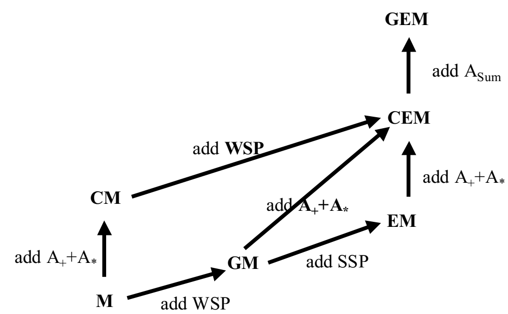
L1: O xy ⇒ P (x*y)x

0	O xy ⇒ (∃z) z=x*y	A_{sum}
1.	O xy	ass
2.	(∃z) z=x*y	0,1 MP
3.	z=x*y	
4.	(∃u)(sum(xyu) & (v)(sum(xyv) ⇒ u=v) & u=z)	3 Dι
5.	sum(xyu) & (v)(sum(xyv) ⇒ u=v) & u=z	
6.	sum(xyu)	5 simp
7.	(w)(P wu ⇔ (P wx & P wy))	6 D_{sum}
8.	P uu ⇔ (P ux & P uy)	7 UI
9.	P ux & P uy	M1,8 MP
10.	P ux	9 simp
11.	u = z	5 simp
12.	u=z & z= x*y	11,3 conj
13.	u=x*y	Tr of =
14.	P (x*y)x	10,13 Id
15.	O xy ⇒ P (x*y)x	1-15 CP

L2 ¬ P xy ⇒ (O xy ⇒ (∃z)(P zx & ¬ O zy))

1.	¬ P xy	ass
2.	O xy	ass
3.	O yx	2, Th MP
4.	P (y*x)y	3, L1 MP
5.	P (x*y)x	2, L1 MP
6.	x=(y*x)	ass
7.	P xy	4,6 Id
8.	¬ P xy & P xy	1,7 conj
9.	¬ x=(y*x)	6-8 IP
10.	P (x*y)x & ¬ x=(y*x)	5,9 conj
11.	PP (x*y)x	10, L6 MP
12.	(∃z)(PP zx & ¬O z(x*y))	11, WSP MP
13.	PP zx & ¬O z(x*y)	
14.	PP zx	13 simp
15.	P zx	14, Th, MP
16.	O zy ⇒ O z(x*y)	15,L4 MP
17.	¬O z (x*y)	13 simp
18.	¬ O zy	16, 17 MT
19.	P zx & ¬ O zy	15, 18 conj
20.	(∃z)(P zx & ¬ O zy)	20 EG, 13-21 EI
21.	O xy ⇒ (∃z)(P zx & ¬ O zy)	2-21 CP
22.	¬ P xy ⇒ (O xy ⇒ (∃z)(P zx & ¬ O zy))	1-22 CP

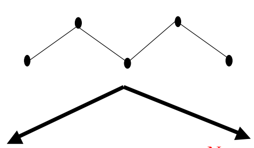




CEM vs. GEM

GEM is strictly stronger than CEM

- Obvious in infinite domains
- But also in finite domains

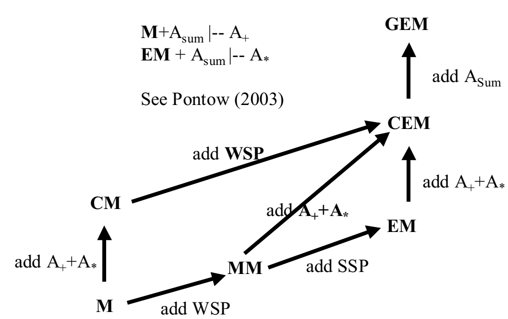


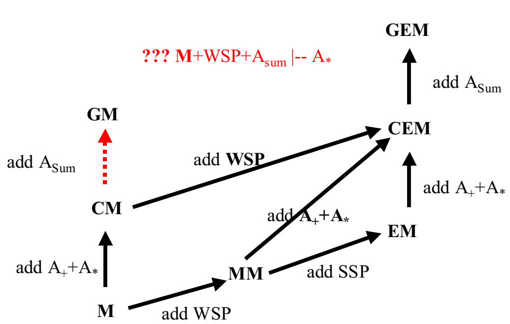
Is a model of CEM

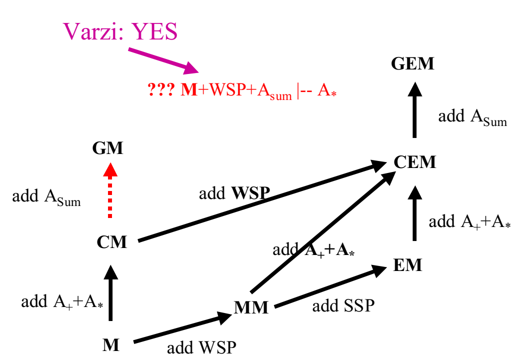
Not a model of GEM
the Universe is definable
as a sum: U = Sum x [P xx]

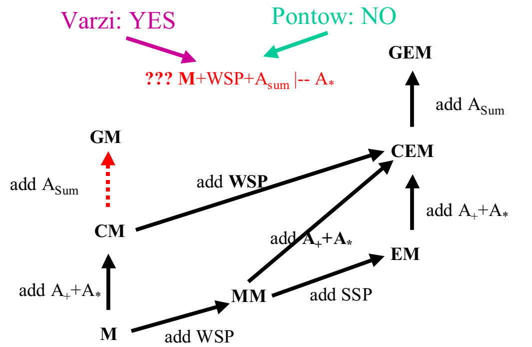
M+A_{sum} |-- A_+
EM + A_{sum} |-- A_*

See Pontow (2003)



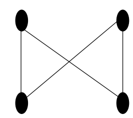






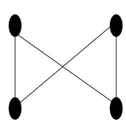
Pontow's counter model

- In order to show that A_* is not derivable from GM = **M**+WSP+A_{sum}
  M+WSP+A_{sum} NOT |-- A_*
- He gives a model that
  - satisfies M, WSP, A_{sum}
  - and does NOT satisfy A_*
  - and does not satisfy SSP

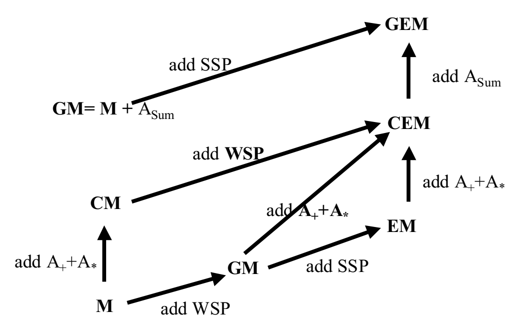


- Complicated: model has to satisfy A_{Sum} for arbitrary formulas ϕ

Assignment



Show that this structure is not a model for A_*



Summary

Ground mereology - M

- Axioms
  - M1 P xx
  - M2 P xy & P yx ⇒ x = y
  - M3 P xy & P yz ⇒ P xz
- Defined relations:
  - Overlap
  - Underlap
  - Proper part

Extending ground mereology

- Adding Principles asserting the existence of entities given the existence of other entities
  - Whenever an entity has one proper part then it has more than one proper part
  - Given two overlapping entities then there exists an entity which is the product of them and given two entities then there exists an entity which is the sum of them
  - Given a set of entities then there exists an entity that is the sum of the entities in that set

Whenever an entity has one proper part then it has more than one proper part

- WSP
  - $PP\ xy \Rightarrow (\exists z)(P\ zy \ \&\ \neg O\ zx)$
- PPP
  - $((\exists z)PP\ zx \ \&\ (\forall z)(PP\ zx \Rightarrow PP\ zy)) \Rightarrow P\ xy$
- SSP
  - $\neg P\ xy \Rightarrow (\exists z)(P\ zx \ \&\ \neg O\ zy)$
- RP
  - $\neg P\ xy \Rightarrow (\exists z)(z = x\text{-}y)$

Extensionality (relation to identity)

Of parthood

$(z)(P\ zx \Leftrightarrow P\ zy) \Leftrightarrow x = y$

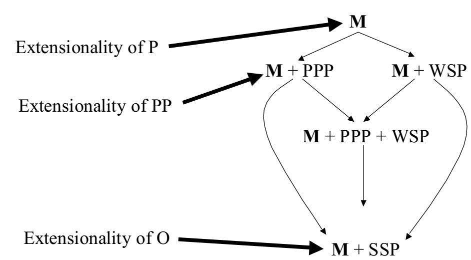
Of proper parthood

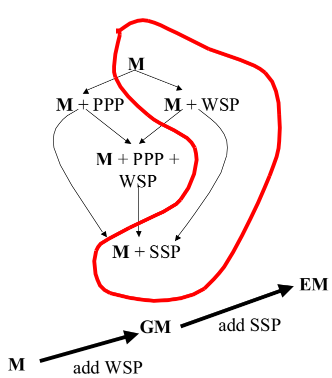
$(\exists z)PP\ zx \ \&\ (z)(PP\ zx \Leftrightarrow PP\ zy) \Leftrightarrow x = y$

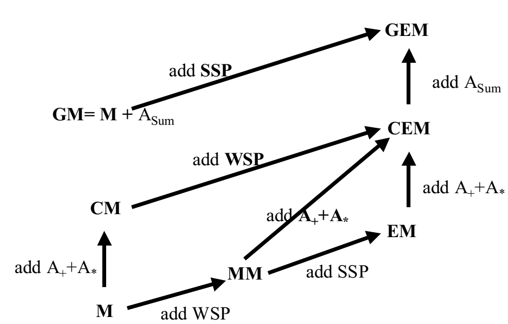
Of overlap

$(z)(O\ zx \Leftrightarrow O\ zy) \Leftrightarrow x = y$

Hierarchy of theories



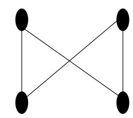




Assignments

- prove that SSP implies RP
  M+SSP |-- $(\exists z)(\forall w)(P\ wz \Leftrightarrow P\ wx\ \&\ \neg O\ wy)$
  or show that this is impossible
- Prove the uniqueness of arbitrary sums (assuming extensionality of O):
  z_1 Sum x φ & z_2 Sum x $\varphi \Rightarrow z_1 = z_2$
- prove
  - $\neg P\ xy \Rightarrow (\neg O\ xy \Rightarrow (\exists z)(P\ zx\ \&\ \neg O\ zy))$
  - $P\ zx \Rightarrow (O\ zy \Rightarrow O\ z(x*y))$
  - **M**+WSP+A_* |-- $\neg P\ xy \Rightarrow (\exists z)(P\ zx\ \&\ \neg O\ zy)$

Assignments (2)



Show that this structure is not a model for A_*