Mereology 6

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Atomistic and atomless mereologies

Atoms

- An atom is an entity with no proper parts
- Definition: $A\,x \equiv \neg(\exists y)\ PP\,yx$
- Questions
  - Are there atoms?
  - If yes is everything *entirely* made up of atoms?
  - Does *everything* comprise at *least of some* atoms?
  - Is *everything* made up of *atomless gunk*?

Axioms of different strength and character are added to the mereology at hand

Mereology is neutral

- All options are logically compatible with mereology developed so far
- Principles regarding atomism can be added to mereology at any level:
  M, MM, EM, CM, CEM, GEM
- Principles of atomicity and atomlessness themselves are *mutually incompatible*
- Need to be added *in separation* to mereology

Atomlessness

- There are no atoms
- Everything made up of atomless gunk
- $\neg A\ x$

Atomicity

Weak atomicity and atomicity

- Weak atomicity
  - There are atoms through not everything needs to have a complete atomic decomposition
  - AT0 $(\exists x)\ Ax$
  - AT0 ensures that there is at least one atom
- Atomicity
  - every entity has an atom as part
  - AT1: $(\exists y)(Ay \ \&\ P\ yx)$

Atomic essentialism (AT2)

- Comes in two equivalent versions
  - AT2(a): $\neg P\ xy \Rightarrow (\exists z)(A\ z\ \&\ P\ zx\ \&\ \neg P\ zy)$
    (Atomic version of SSP)
    $\neg P\ xy \Rightarrow (\exists z)(P\ zx\ \&\ \neg O\ zy)$ (SSP)
  - AT2(b): $(z)(A\ z \Rightarrow (P\ zx \Rightarrow P\ zy)) \Rightarrow P\ xy$
- Assignment:
  - prove the equivalence of AT2(a) and AT2(b)
  - prove that AT2 implies SSP, I.e., M+AT2 |- SSP

Equivalence of AT2(a) and AT2(b)

- Use the following logical equivalences:
  - Trans
  - QN
  - DN
  - DeM
  - Impl

AT2(a) ⇔ AT2(b)

1.	¬P xy ⇒ (∃z)(A z & P zx & ¬P zy)	ass
2.	¬(∃z)(A z & P zx & ¬P zy) ⇒ ¬¬P xy	1 trans
3.	¬(∃z)(A z & P zx & ¬P zy) ⇒ P xy	2 DN
4.	(z)¬(A z & P zx & ¬P zy) ⇒ P xy	3 QN
5.	(z)(¬A z or ¬ (P zx & ¬P zy)) ⇒ P xy	4 DeM
6.	(z)(¬A z or (P zx ⇒P zy)) ⇒ P xy	5 Imp
7.	(z)(A z ⇒ (P zx ⇒P zy)) ⇒ P xy	6 Imp

M+AT2 |- ¬P xy ⇒ (∃z)(P zx & ¬O xy)

0.	P xy⇒PP xy or x=y	
1.	¬P xy	ass
2.	(∃z)(Az & P zx & ¬P zy)	1 AT2 MP
3.	Az & P zx & ¬P zy	
4.	O zy	ass
5.	(∃u)(P uz & P uy)	4 D_O
6.	P uz & P uy	
7.	P uz	6 simp
8.	PP uz or u=z	7,0 MP
9.	u=z	ass
10.	¬P uy	(3 simp), 9 Id
11.	P uy & ¬P uy	(6 simp), 10 conj
12.	¬ (u=z)	9-11 IP
13.	PP uz	8,12 DS
14.	(∃u) PP uz	13 EG
15.	¬ A z	14 D_A
16.	A z & ¬ A z	(3 simp), 15 conj
17.	¬ O zy	4-16 IP
18.	P zx & ¬ O zy	(3 simp), 17 conj
19.	(∃u) (P zx & ¬ O zy)	18 EG
20.	¬P xy ⇒ (∃u) (P zx & ¬ O zy)	1-19 CP

Atomic essentialism (AT2)

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  - prove that AT2 implies SSP, I.e., M+AT2 |- SSP

Atomic essentialism (2)

Given AT2 we can prove that

- Two things are identical iff they have the same atoms as parts
- $x=y \Leftrightarrow (z)(A\ z \Rightarrow P\ zx \Leftrightarrow P\ zy)$
- Very strong:
  - For identity it is sufficient to look at the atoms.
  - Other parts do not matter

Assignment

- Prove that
  $x=y \Leftrightarrow (z)(A\ z \Rightarrow P\ zx \Leftrightarrow P\ zy)$
  follows from AT2
- i.e.,
  $M+\ AT2 \vdash x=y \Leftrightarrow (z)(A\ z \Rightarrow P\ zx \Leftrightarrow P\ zy)$
- Hint:
  - it is easier to use AT2(b)
  - Use $(P \Rightarrow (Q \,\&\, R)) \Rightarrow ((P \Rightarrow Q)\&(P \Rightarrow R))$

$M+\ AT2 \vdash (z)(A\ z \Rightarrow P\ zx \Leftrightarrow P\ zy) \Rightarrow x=y$

0.	$(P \Rightarrow (Q \,\&\, R)) \Rightarrow ((P \Rightarrow Q)\&(P \Rightarrow R))$	
1.	$(z)(A\ z \Rightarrow P\ zx \Leftrightarrow P\ zy)$	ass
2.	$A\ z \Rightarrow P\ zx \Leftrightarrow P\ zy$	1 UI
3.	$A\ z \Rightarrow ((P\ zx \Rightarrow P\ zy)\&\ (P\ zy \Rightarrow P\ zx))$	2 Eq
4.	$(A\ z \Rightarrow (P\ zx \Rightarrow P\ zy))\ \&\ (A\ z \Rightarrow (P\ zy \Rightarrow P\ zx))$	3, 0 MP
5.	$(A\ z \Rightarrow (P\ zx \Rightarrow P\ zy))$	4 simp
6.	$(z)(A\ z \Rightarrow (P\ zx \Rightarrow P\ zy))$	5 UG
7.	$P\ xy$	6,AT2 MP
8.	$(A\ z \Rightarrow (P\ zy \Rightarrow P\ zx))$	4 simp
9.	$(z)(A\ z \Rightarrow (P\ zy \Rightarrow P\ zx))$	8 UG
10.	$P\ yx$	9,AT2 MP
11.	$P\ xy\ \&\ P\ yx$	7,10 conj
12.	$x=y$	11, M2 MP
13.	$(z)(A\ z \Rightarrow P\ zx \Leftrightarrow P\ zy) \Rightarrow x=y$	1-12 CP

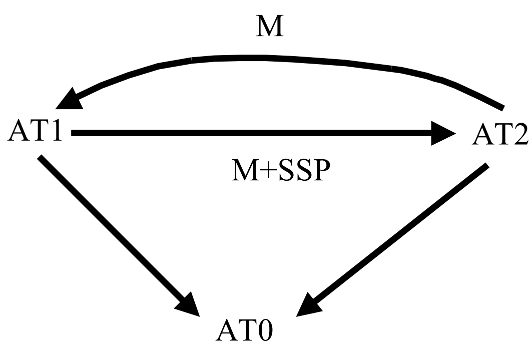
M+ AT2 |-- $x=y \Rightarrow (z)(A\,z \Rightarrow P\,zx \Leftrightarrow P\,zy)$

1.	$x=y$	ass
2.	Az	ass
3.	$P\,zx$	ass
4.	$P\,zy$	3,1 Id
5.	$P\,zx \Rightarrow P\,zy$	3-4 CP
6.	$P\,zy$	ass
7.	$P\,zx$	6,1 Id
8.	$P\,zy \Rightarrow P\,zx$	6-7 CP
9.	$P\,zx \Leftrightarrow P\,zy$	(5,8 conj) Eq
10.	$Az \Rightarrow (P\,zx \Leftrightarrow P\,zy)$	2-9 CP
11.	$(z)(Az \Rightarrow (P\,zx \Leftrightarrow P\,zy))$	10 UG
12.	$x=y \Rightarrow (z)(A\,z \Rightarrow P\,zx \Leftrightarrow P\,zy)$	1-11 CP

- We proved that
  $x=y \Leftrightarrow (z)(A\,z \Rightarrow P\,zx \Leftrightarrow P\,zy)$
  follows from AT2
- i.e.,
  M+ AT2 |-- $x=y \Leftrightarrow (z)(A\,z \Rightarrow P\,zx \Leftrightarrow P\,zy)$

Relations between AT0,AT1,AT2

Masolo & Vieu 01



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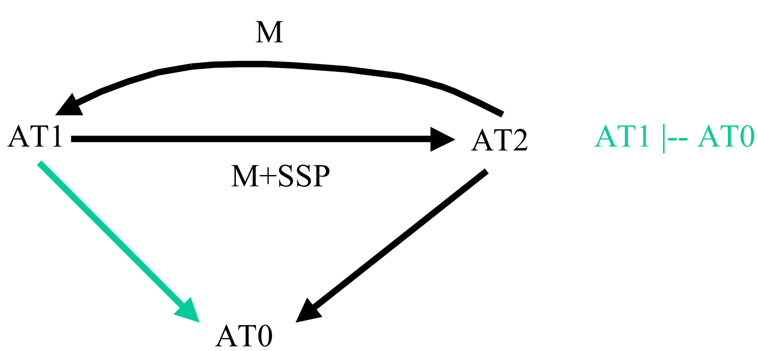

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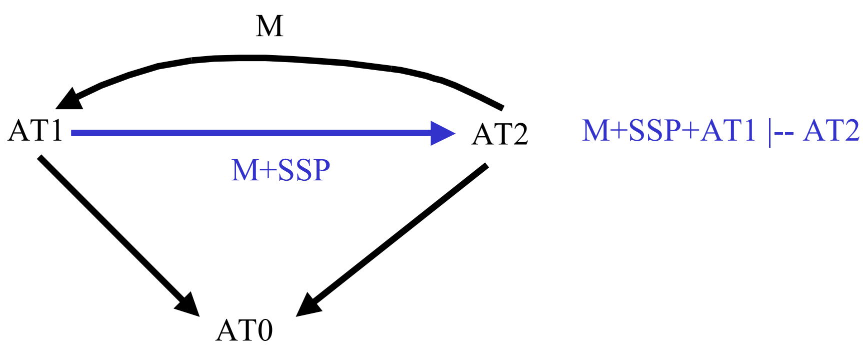

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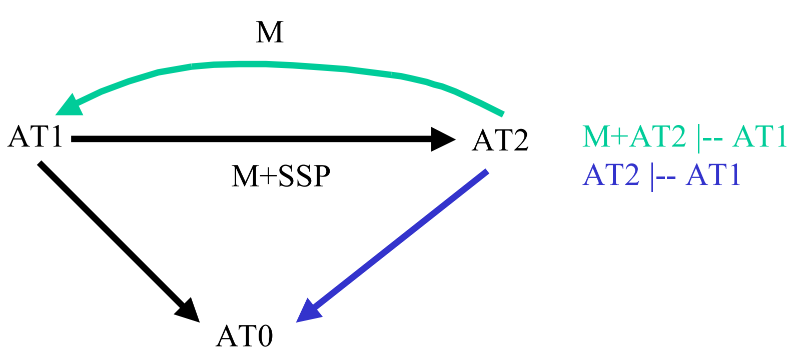

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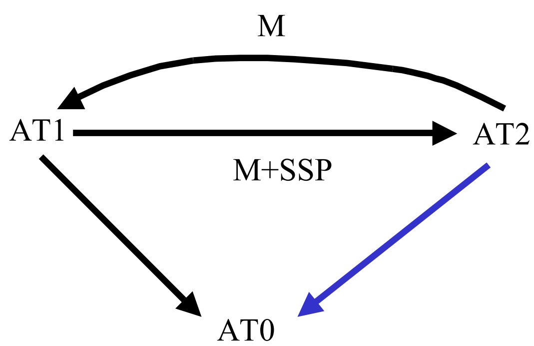
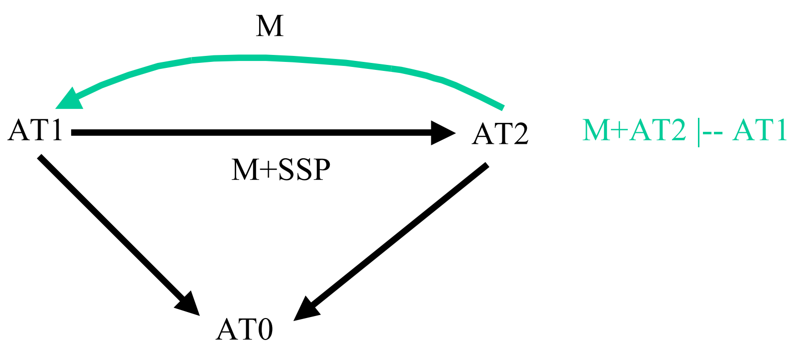


AT2 |-- AT1

M+AT2 |-- $(\exists x)Az$

0.	$\neg P\,xy \Rightarrow (\exists z)(A\,z \;\&\; P\,zx \;\&\; \neg P\,zy)$	
1.	$\neg(\exists x)Ax$	ass
2.	$(x)\,\neg\,Ax$	1 QN
3.	$\neg\,Ax$	2 UI
4.	$\neg\neg\,(\exists y)PP\,yx$	3 D_A
5.	$(\exists y)PP\,yx$	4 DeM
6.	$PP\,yx$	
7.	$P\,yx \;\&\; \neg(x=y)$	6 D_{PP}
8.	$\neg(x=y)$	7 simp
9.	$\neg(P\,xy \;\&\; P\,yx)$	8, M2 MT
10.	$\neg P\,xy$ or $\neg\,P\,yx$	9 DeM
11.	$\neg P\,xy$	(7 simp),10 DS
12.	$(\exists z)(A\,z \;\&\; P\,zx \;\&\; \neg P\,zy)$	11, 0 MP
13.	$A\,z \;\&\; P\,zx \;\&\; \neg P\,zy$	
14.	Az	13 simp
15.	$(\exists x)\,Ax$	14 EG
16.	$(\exists x)\,Ax \;\&\; \neg(\exists x)Ax$	15, 1 conj
17.	$(\exists x)\,Ax$	1-16 IP

Relations between AT0,AT1,AT2

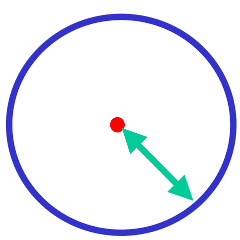


M+AT2 |-- (∃z)(Az & P zx)

0.	(z)((A z &P zx) ⇒P zy)) ⇒ P xy	
1.	¬(∃z)(Az & P zx)	ass
2.	(z) ¬(Az & P zx)	1 QN
3.	(z)(¬ P zx or ¬Az)	2 DeM
4.	¬ P xx or ¬Ax	3 UI
5.	¬Ax	4, M1 DS
6.	¬ ¬(∃y)PP yx	5 D_A
7.	(∃y)PP yx	6 DN
8.	P yx & ¬ (y=x)	7 D_{PP}
9.	A z &P zx	ass
10.	¬ P zx or ¬Az	3 UI
11.	¬Az	(9 simp),10 DS
12.	Az & ¬Az	(9 simp), 11 conj
13.	(Az & ¬Az) or P zy	12 add
14.	(Az or P zy) & (¬Az or P zy)	13 dist
15.	(¬ Az ⇒ P zy) & (Az ⇒P zy)	14 Imp
16.	¬ Az or Az	ExMiddle
17.	P zy or Pzy	15, 16 CD
18.	P zy	17 taut
19.	(A z &P zx) ⇒ P zy	9-18 CP
20.	(z)((A z &P zx) ⇒ P zy)	19 UG
21.	P xy	20, 0 MP
22.	P yx & P xy	(8 simp), 21 conj
23.	x=y	22, M2 MP
24.	¬ (y=x) & x=y	(8 simp) 23 conj
25.	(∃z)(Az & P zx)	1-24 IP

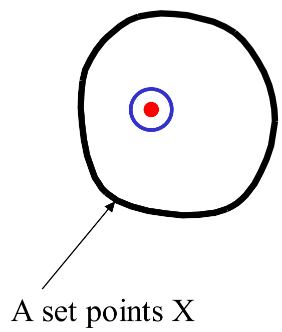
Point set topology

Neighborhoods of points



- Assume the set of points or the Euclidian plane
- A *neighborhood* of a Point P is a disk of radius ν with center P

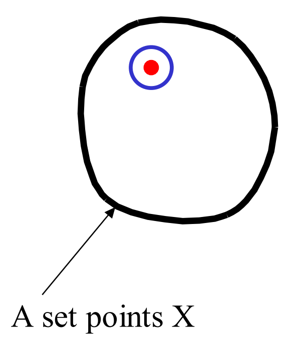
Sets and neighborhoods



Classification of points with respect to X

- Interior points wrt. X:
  - Points which have a neighborhood which *contains only members of X*

Sets and neighborhoods



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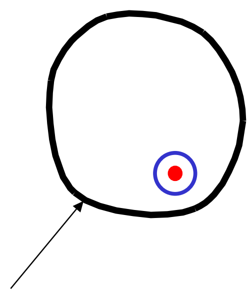

A set points X

Classification of points with respect to X

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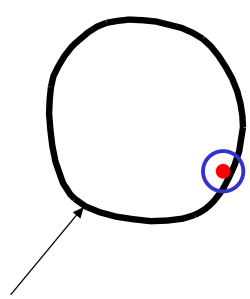
Sets and neighborhoods

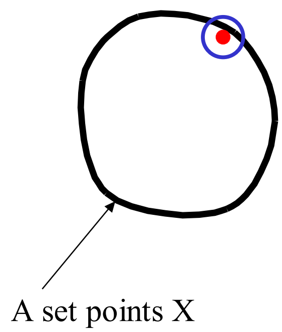


A set points X

Boundary points wrt. X :

- Points which have a neighborhood which
  - contains members of X
  - And contains non-members of X

Sets and neighborhoods



Boundary points:

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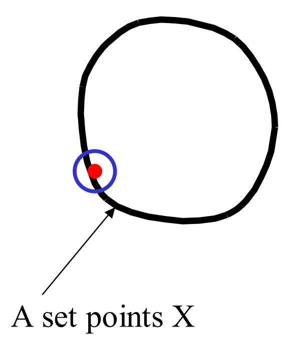
Sets and neighborhoods



Boundary points:

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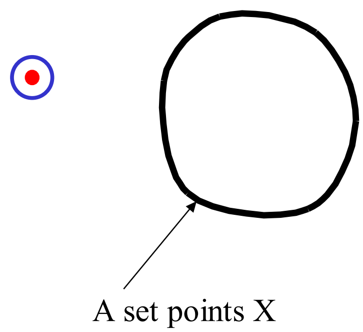
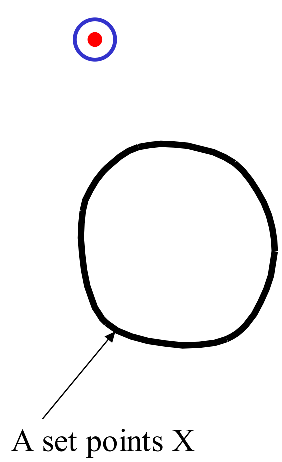
Sets and neighborhoods



Exterior points wrt X:

- Points which have a neighborhood which soes NOTcontain members of X

Sets and neighborhoods



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Sets and neighborhoods

Exterior points wrt X:

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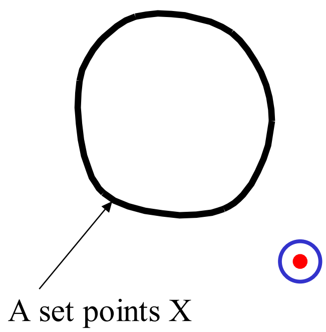
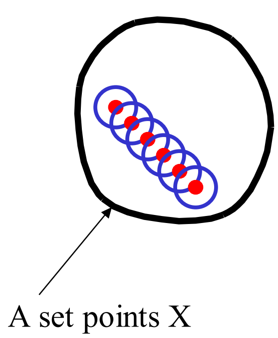
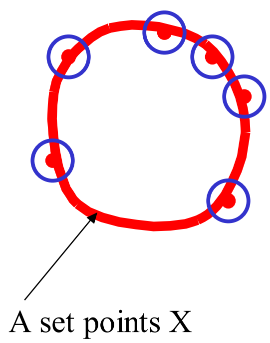


The interior of a Set



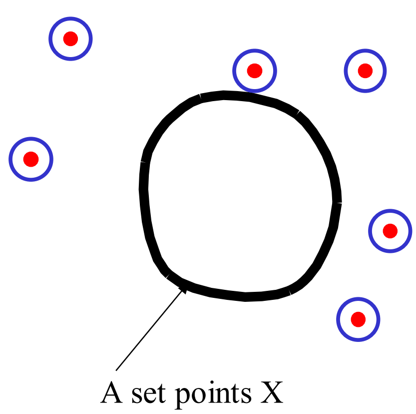
- X is a set
- i(X) the interior of X
- Is the set which contains all of Xs interior points
- $i(X) \subseteq X$

The boundary of a Set



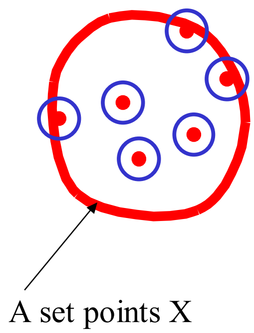
- X is a set
- b(X) the boundary of X
- Is the set which contains all of Xs boundary points
- b(X) contains some points which are not elements of X

The exterior of a Set



- X is a set
- e(X) the exterior of X
- Is the set which contains all of Xs exterior points
- e(X) contains points only which are not elements of X

The closure of a Set



- X is a set
- cl(X) the closure of X
- Is the set which contains all of Xs *interior and boundary* points
- cl(X) contains some points which are not elements of X

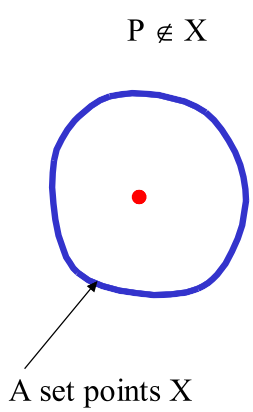
Relationships between interior, boundary, closure, and exterior

- $i(X) \subseteq X$
- $i(X) \cap b(X) = \varnothing$
- $cl(X)= i(X) \cup b(X)$
- Let P be the points of the plane and $X \subseteq P$ then we have
  $P = i(X) \cup b(X) \cup e(X)$

Regular open sets

- A set is *open* iff it contains only *interior points*
- A set is *regular open* iff it is identical to the *interior of its closure*
- *RO X ≡ X=I(cl(X))*

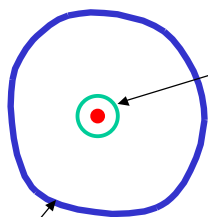
Non-regular open sets



- Let X be a set without the point P
- Let i(X) be the interior of X
- i(X) is open
- i(X) is NOT regular

Non-regular open sets

P $\notin$ X

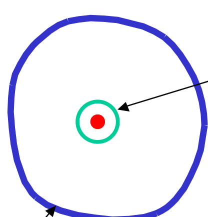


P is a *boundary* point of X

A set points X

Non-regular open sets

P $\notin$ X



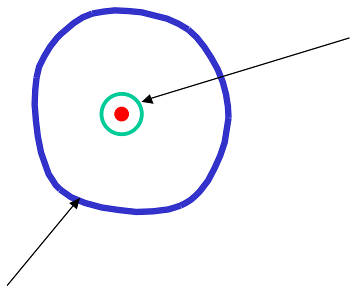
P is a *boundary* point of X

P $\in$ cl(X)

A set points X

Non-regular open sets

P $\notin$ X



P is a *boundary* point of X

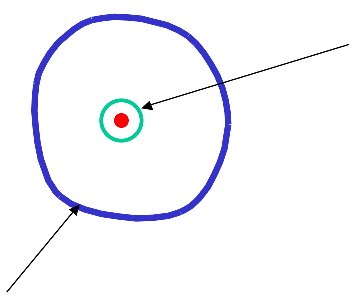
P $\in$ cl(X)

P is *NOT* a boundary point of the closure of X

A set points X

Non-regular open sets

P $\notin$ X



P is a *boundary* point of X

P $\in$ cl(X)

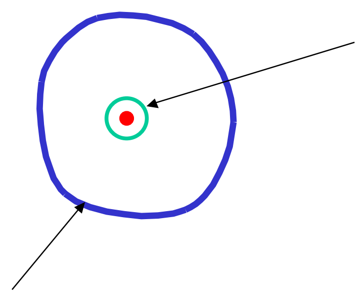
P is *NOT* a boundary point of the closure of X

P $\in$ i(cl(X))

A set points X

Non-regular open sets

P $\notin$ X



P is a *boundary* point of X

P $\in$ cl(X)

P is *NOT* a boundary point of the closure of X

A set points X

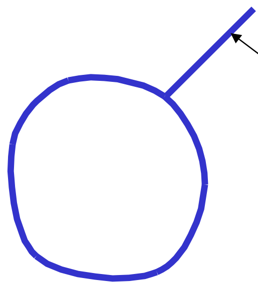
P $\in$ i(cl(X))

i(X) $\neq$ i(cl(X))
i(X) is not regular open

Regular closed sets

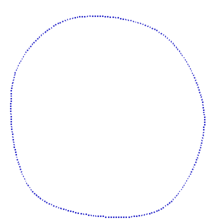
- A set is *closed* iff it contains only *interior and boundary points*
- A set is *regular closed* iff it is identical to the *closure of its interior*
- *RC X* $\equiv$ *X=cl(i(X))*

Non-regular closed sets



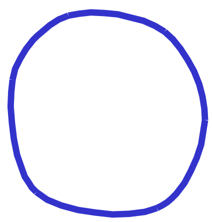
X is a closed set of points
which has a
boundary spike without interior

Non-regular closed sets



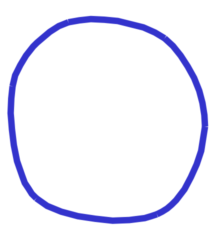
The interior of X, i(X), does not
contain boundary points –
The spike is gone

Non-regular closed sets



The closure of the interior of X, cl(i(X)), contains only boundary points which are close to its interior

Non-regular closed sets



The closure of the interior of X, cl(i(X)), contains only boundary points which ore close to its interior

The spike remains gone

$X \neq cl(i(X))$

X is NOT regular closed

Regular sets are topologically nice and regular – no lower dimensional holes or spikes

Topologies

- A set Z with a system of (regular) open sets $\boldsymbol{Z}$ such that
  - $Z \in \boldsymbol{Z}$
  - $\emptyset \in \boldsymbol{Z}$
  - $\boldsymbol{Z}$ is closed under finite (regularized) unions
    If $X \in \boldsymbol{Z}$ & $Y \in \boldsymbol{Z}$ then $X \cup Y \in \boldsymbol{Z}$
  - $\boldsymbol{Z}$ is closed under arbitrary (regularized) intersections

    If $U \subseteq \boldsymbol{Z}$ then $\bigcap_{Y \in U} \in \boldsymbol{Z}$

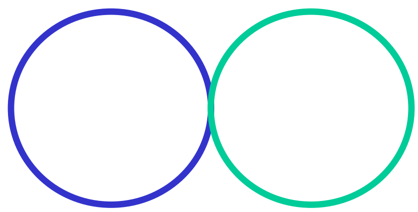
Topologies

- A set Z with a system of (regular) closed sets $\mathcal{Z}$ such that
  - $Z \in \boldsymbol{Z}$
  - $\emptyset \in \boldsymbol{Z}$
  - $\boldsymbol{Z}$ is closed under finite (regularized) intersections
    If $X \in \boldsymbol{Z}$ & $Y \in \boldsymbol{Z}$ then $X \cap Y \in \boldsymbol{Z}$
  - $\boldsymbol{Z}$ is closed under arbitrary (regularized) unions
    If $U \subseteq \boldsymbol{Z}$ then $\bigcup_{Y \in U} \in \boldsymbol{Z}$

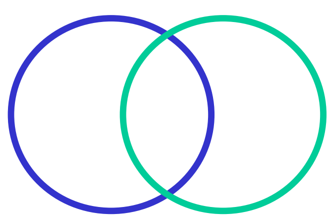
Connectedness

- Two sets X and Y are connected iff
  - X intersects the closure of Y or Y intersects the closure of X
  - $X \cap cl(Y) \neq \emptyset$ or $Y \cap cl(X) \neq \emptyset$
- Important:
  - For connectedness the interiors do NOT need to overlap
  - Connected sets do NOT need to share interior points
- Regular closed sets: connected if they share at least one point of their closures

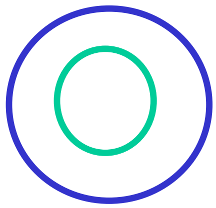
X is connected to Y



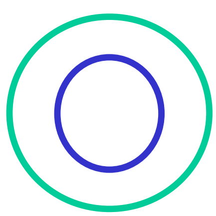
X is connected to Y



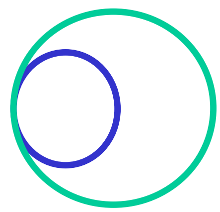
X is connected to Y



X is connected to Y



X is connected to Y



Mereotopology –
the formal theory of parthood and connectedness

Need for topology (Varzi)

- Mereological reasoning cannot do justice to the notion of a *whole*
- Distinction between
  - one-piece, self-connected wholes like stone, whistle
  - Scattered entities made up of several disconnected parts like a broken glass, a bikini, a sum of two disjoint cats

  cannot be expressed in mereology

Need for topology (2)

- In GEM for any connection of parts there is in principle a complete whole:
  the mereological sum
- There is no way, within mereology, to draw a distinction between 'good' and 'bad' sums – between
  - Integral wholes and
  - Scattered sums of disperate entities

Way out:

- Mereological account must be supplemented with a topological machinery of some sort
- Mereology a *part-of* theory
- Mereotopology is *part-whole* theory
- Add a primitive binary relation $C\,xy$ interpreted as
  x is-topologically-connected-to y

Why not using point set topology?

- Point set topology is based on set theory
- If we found our topological theory on sets then we import all philosophical problems of sets:
  - Need for (minimal) elements
  - How can infinitely many non-extended points constitute extended entities?
  - See Barry's 'topological foundations of cognitive science' paper for more arguments

BUT !!

- Using point set topology at the level of models is gooooood!
- Helps us
  - To better understand our theories
  - To find proofs and counter models

Ground topology

Ground mereology - **M**

- Axioms
  - M1 P xx
  - M2 P xy & P yx $\Rightarrow$ x = y
  - M3 P xy & P yz $\Rightarrow$ P xz
- Defined relations:
  - Overlap
  - Underlap
  - Proper part

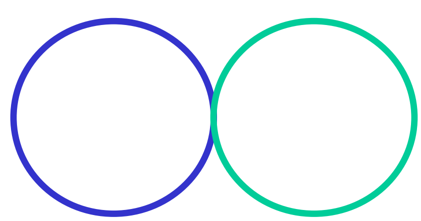
Interpretation of ground mereology in a topological space T=(Z,***Z***)

- P xy is interpreted as $i(X) \subseteq i(Y)$ with X,Y $\in$ ***Z***
- From the interpretation of P it follows that O xy holds if and only X and Y share at least one *interior* point

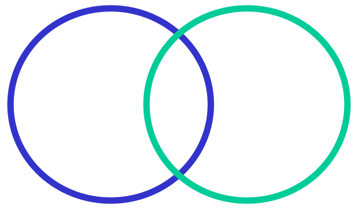
Ground topology

- Primitive relation *C xy*
- Interpretation *x* is-connected-to *y*
- If *x* and *y* are interpreted of regular closed sets of some topological space then *C xy* is interpreted as the relation which holds iff the closures of X and Y share at least one point

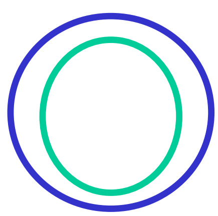
C xy



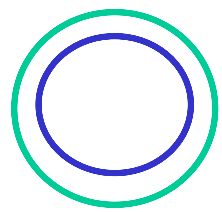
C xy



C xy



$C\ xy$



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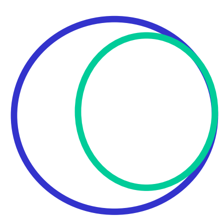
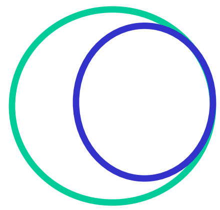


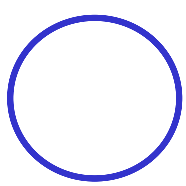
C xy



Axioms of ground topology

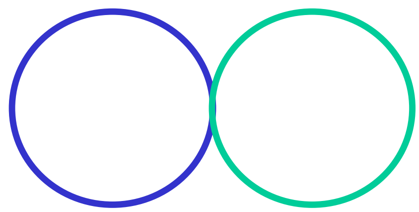
- C1: C is reflexive

  C xx



Axioms of ground topology

- C1: C is reflexive
  C xx
- C2: C is symmetric
  C xy ⇒ C yx



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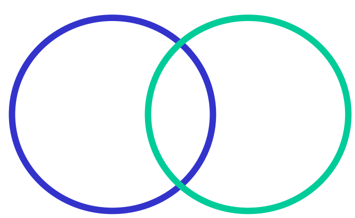

Axioms of ground topology

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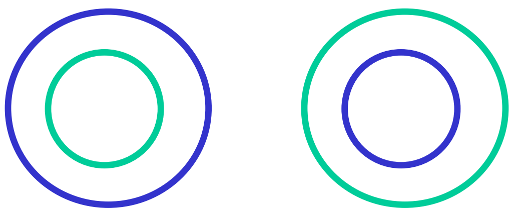
  $C\ xx$
- C2: C is symmetric
  $C\ xy \Rightarrow C\ yx$



Axioms of ground topology

- C1: C is reflexive
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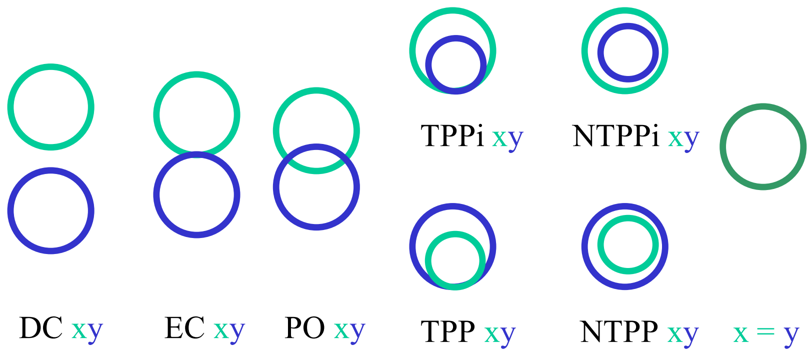
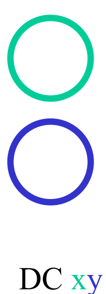
- C2: C is symmetric
  $C\ xy \Rightarrow C\ yx$
- C3: relation between P and C
  if x is a part of y then everything that is connected to x is also connected to y
  $P\ xy \Rightarrow (z)(C\ zx \Rightarrow C\ zy)$

The RCC8-relations

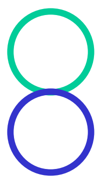


The RCC8-relations



$DC\ xy \equiv \neg C\ xy$

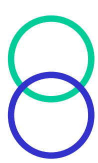
The RCC8-relations



EC xy ≡ C xy & ¬O xy

EC xy

The RCC8-relations

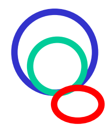


PO xy ≡ O xy & ≡¬P xy & ≡¬P xy

PO xy

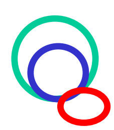
The RCC8-relations

TPP xy ≡ PP xy & (∃z)(EC zx & EC zy)



TPP xy

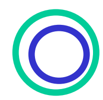
The RCC8-relations



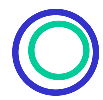
TPPi xy

TPPi xy ≡ PP xy & (∃z)(EC zx & EC zy)

The RCC8-relations

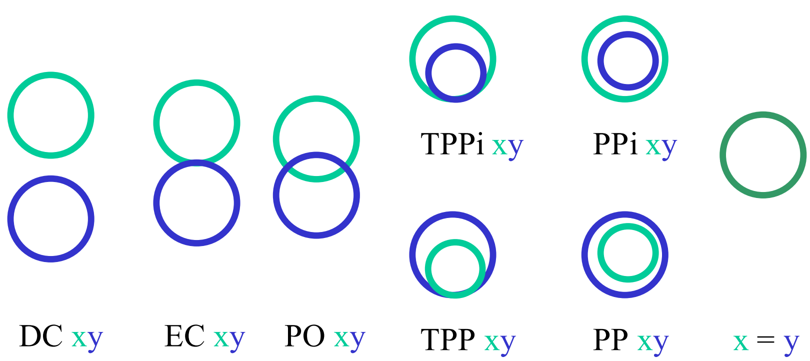
$\text{NTPPi}\ xy \equiv \text{PP}\ yx\ \&\ \neg(\exists z)(\text{EC}\ zx\ \&\ \text{EC}\ zy)$ 

NTPPi xy

$\text{NTPP}\ xy \equiv \text{PP}\ xy\ \&\ \neg(\exists z)(\text{EC}\ zx\ \&\ \text{EC}\ zy)$ 

NTPP xy

The RCC8-relations



Assignments

Prove the following theorems

1. DC, EC, PO are symmetric
   e.g., DC xy ⇒ DC yx
2. (N)TPP and (N)TPPi are asymmetric
   e.g., TPP xy ⇒ ¬TPPi xy

The RCC8 lattice

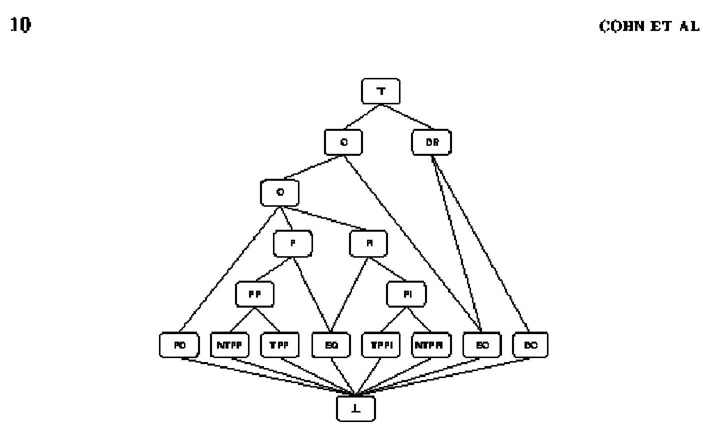
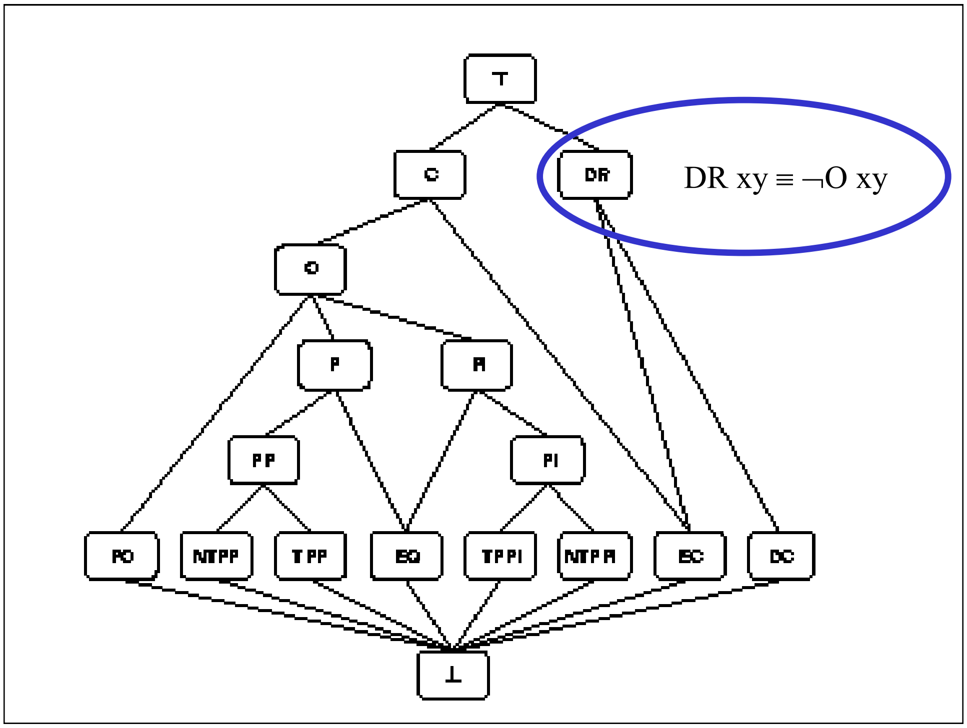
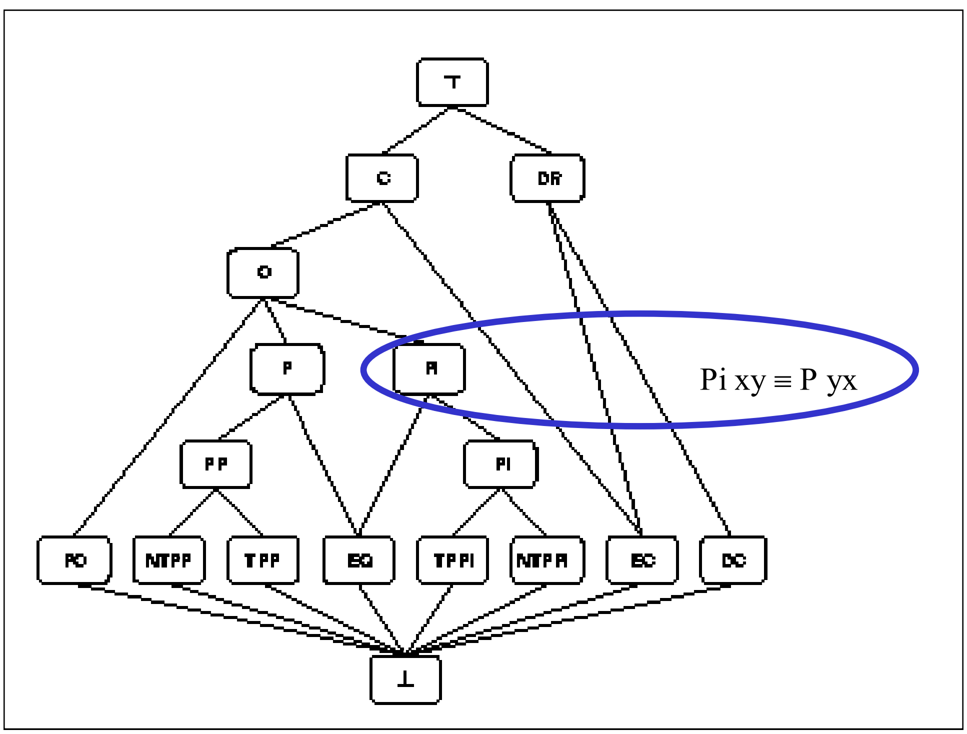
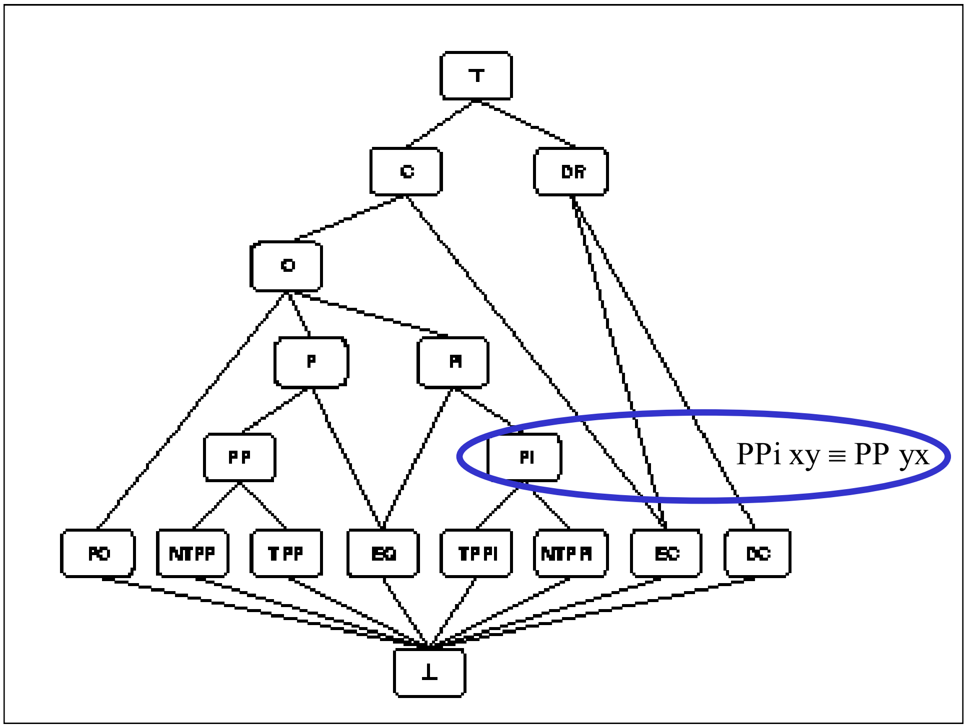
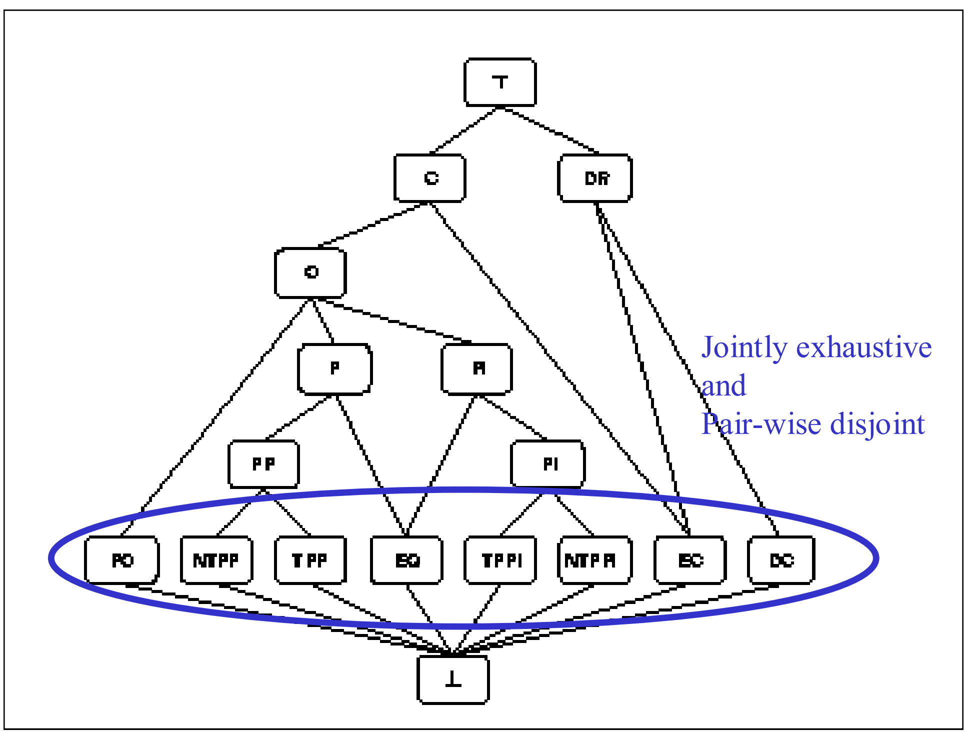


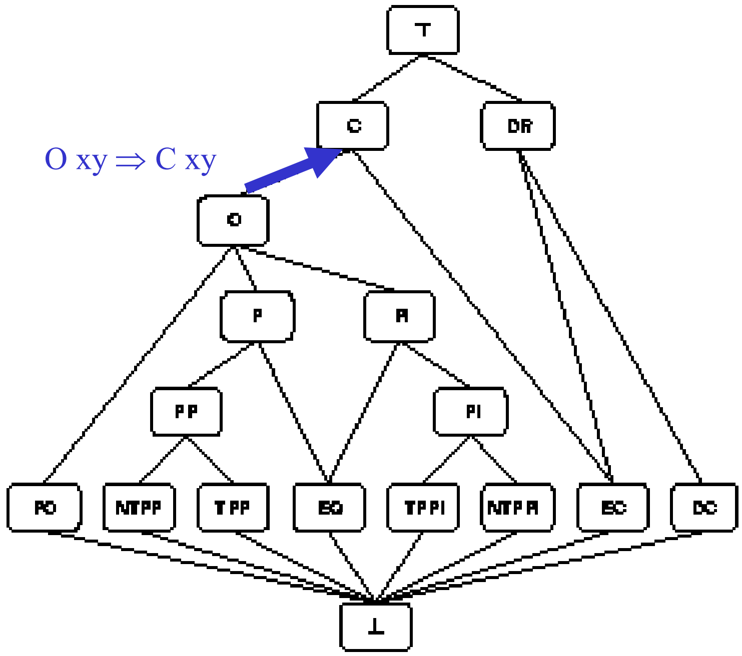
Figure 4. A subsumption lattice of dyadic relations defined in terms of C


⊤
C
DR
DR xy ≡ ¬O xy
O
P
Pi
PP
PI
PO
NTPP
TPP
EQ
TPPI
NTPPI
EC
DC
⊥


⊤
C
DR
O
P
Pi
Pi xy ≡ P yx
PP
PI
PO
NTPP
TPP
EQ
TPPI
NTPPI
EC
DC
⊥


⊤
C
DR
O
P
PI
PP
PPI
PPi xy ≡ PP yx
PO
NTPP
TPP
EQ
TPPI
NTPPI
EC
DC
⊥

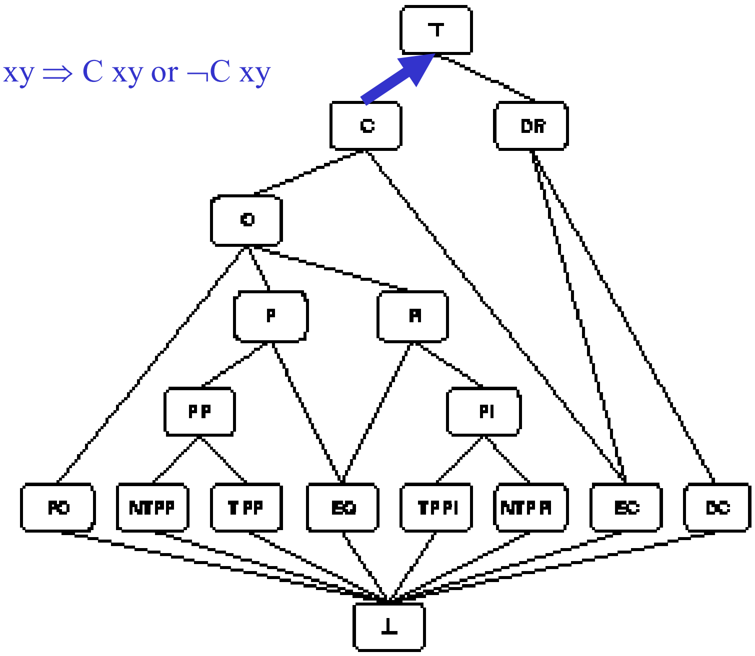

⊤
C
DR
O
P
PI
PP
PPI
Jointly exhaustive
and
Pair-wise disjoint
PO
NTPP
TPP
EQ
TPPI
NTPPI
EC
DC
⊥



$O\ xy \Rightarrow C\ xy$

0.	$P\ xy \Rightarrow (z)(C\ zx \Rightarrow C\ zy)$	C3
1.	$O\ xy$	ass
2.	$(\exists z)(P\ zx\ \&\ P\ zy)$	1 D_O
3.	$P\ zx\ \&\ P\ zy$	
4.	$P\ zx$	3 simp
5.	$(u)(C\ uz \Rightarrow C\ ux)$	4,0 MP
6.	$C\ yz \Rightarrow C\ yx$	5 UI
7.	$P\ zy$	3 simp
8.	$(u)(C\ uz \Rightarrow C\ uy)$	7,0 MP
9.	$C\ zz \Rightarrow C\ zy$	8 UI
10.	$C\ zy$	9, C1 MP
11.	$C\ yz$	10, C2 MP
12.	$C\ yx$	11, 6 MP
13.	$C\ xy$	12, C2 MP
14.	$O\ xy \Rightarrow C\ xy$	1-13 CP

$C\ xy \Rightarrow C\ xy$ or $\neg C\ xy$



More assignments

- Prove the following theorems
  - $PO\ xy \Rightarrow O\ xy$
  - $\neg(PO\ xy\ \&\ NTPP\ xy)$
  - $DC\ xy \Rightarrow DR\ xy$
  - $EC\ xy \Rightarrow DR\ xy$
  - $\neg(EC\ xy\ \&\ DC\ xy)$

  using their definitions and C1-C3

Summary

Weak atomicity and atomicity

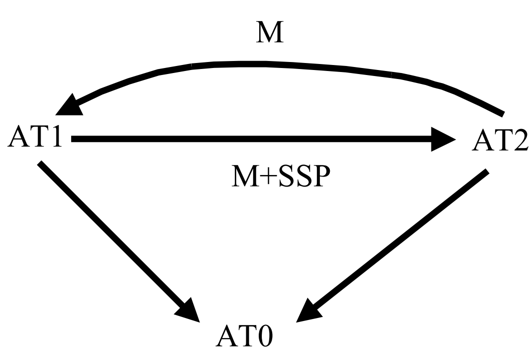
- Weak atomicity
  - There are atoms through not everything needs to have a complete atomic decomposition
  - AT0 $(\exists x)\ Ax$
  - AT0 ensures that there is at least one atom
- Atomicity
  - every entity has an atom as part
  - AT1: $(\exists y)(Ay\ \&\ P\ yx)$

Atomic essentialism (AT2)

- Comes in two equivalent versions
  - AT2(a): $\neg P\ xy \Rightarrow (\exists z)(A\ z\ \&\ P\ zx\ \&\ \neg P\ zy)$
    (Atomic version of SSP)
    $\neg P\ xy \Rightarrow (\exists z)(P\ zx\ \&\ \neg O\ zy)$ (SSP)
  - AT2(b): $(z)(A\ z \Rightarrow (P\ zx \Rightarrow P\ zy)) \Rightarrow P\ xy$
- Assignment:
  - prove the equivalence of AT2(a) and AT2(b)
  - prove that AT2 implies SSP, I.e., M+AT2 |- SSP

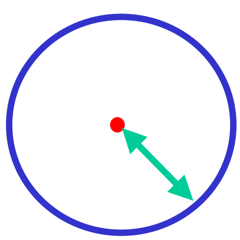
Relations between AT0,AT1,AT2

Masolo & Vieu 01



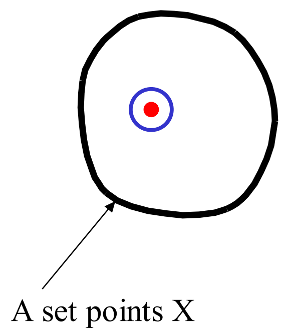
Point set topology

Neighborhoods of points



- Assume the set of points or the Euclidian plane
- A *neighborhood* of a Point P is a disk of radius ν with center P

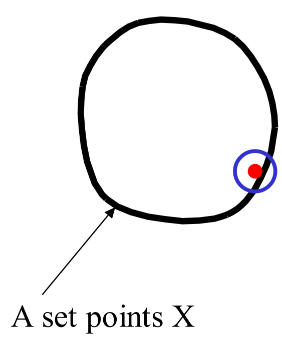
Sets and neighborhoods



Classification of points with respect to X

- Interior points wrt. X:
  - Points which have a neighborhood which *contains only members of X*

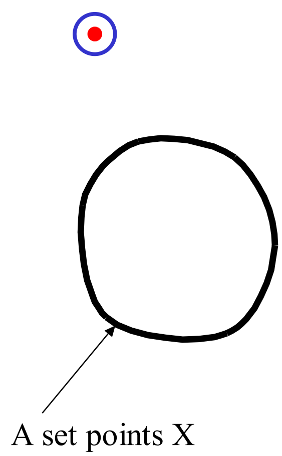
Sets and neighborhoods



Boundary points wrt. X :

- Points which have a neighborhood which
  - contains members of X
  - And contains non-members of X

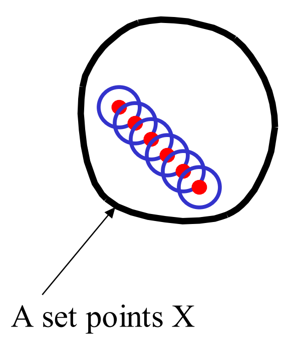
Sets and neighborhoods



Exterior points wrt X:

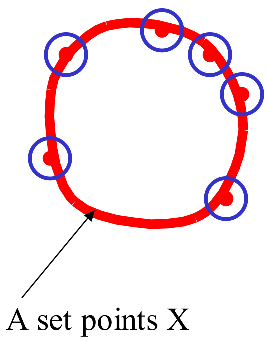
- Points which have a neighborhood which soes NOTcontain members of X

The interior of a Set



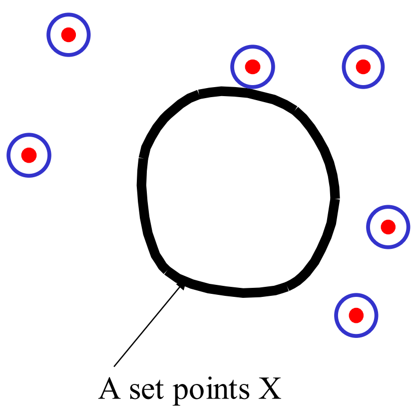
- X is a set
- i(X) the interior of X
- Is the set which contains all of Xs interior points
- $i(X) \subseteq X$

The boundary of a Set



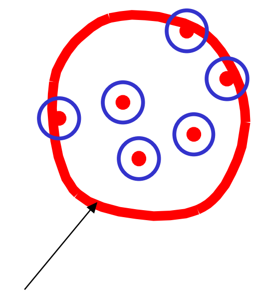
- X is a set
- b(X) the boundary of X
- Is the set which contains all of Xs boundary points
- b(X) contains some points which are not elements of X

The exterior of a Set



- X is a set
- e(X) the exterior of X
- Is the set which contains all of Xs exterior points
- e(X) contains points only which are not elements of X

The closure of a Set



- X is a set
- cl(X) the closure of X
- Is the set which contains all of Xs *interior and boundary* points
- cl(X) contains some points which are not elements of X

Relationships between interior, boundary, closure, and exterior

- $i(X) \subseteq X$
- $i(X) \cap b(X) = \varnothing$
- $cl(X) = i(X) \cup b(X)$
- Let P be the points of the plane and $X \subseteq P$ then we have
  $P = i(X) \cup b(X) \cup e(X)$

Regular open sets

- A set is *open* iff it contains only *interior points*
- A set is *regular open* iff it is identical to the *interior of its closure*
- $RO\ X \equiv X = I(cl(X))$

Regular closed sets

- A set is *closed* iff it contains only *interior and boundary points*
- A set is *regular closed* iff it is identical to the *closure of its interior*
- $RC\ X \equiv X = cl(i(X))$

Topologies

- A set Z with a system of (regular) closed sets *Z* such that
  - $Z \in \boldsymbol{Z}$
  - $\emptyset \in \boldsymbol{Z}$
  - $\boldsymbol{Z}$ is closed under finite (regularized) intersections
    If $X \in \boldsymbol{Z}$ & $Y \in \boldsymbol{Z}$ then $X \cap Y \in \boldsymbol{Z}$
  - $\boldsymbol{Z}$ is closed under arbitrary (regularized) unions
    If $U \subseteq \boldsymbol{Z}$ then $\bigcup_{Y \in U} \in \boldsymbol{Z}$

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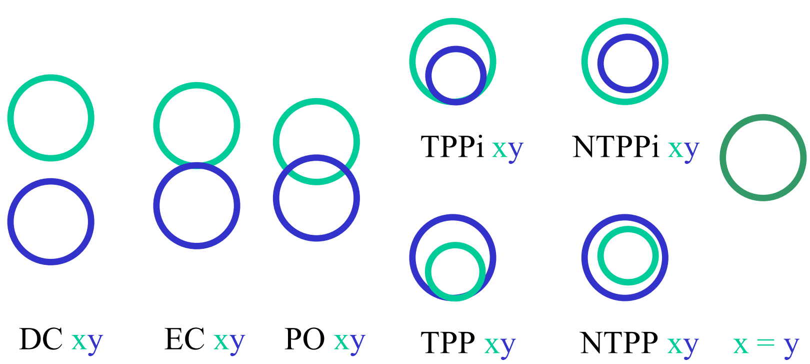
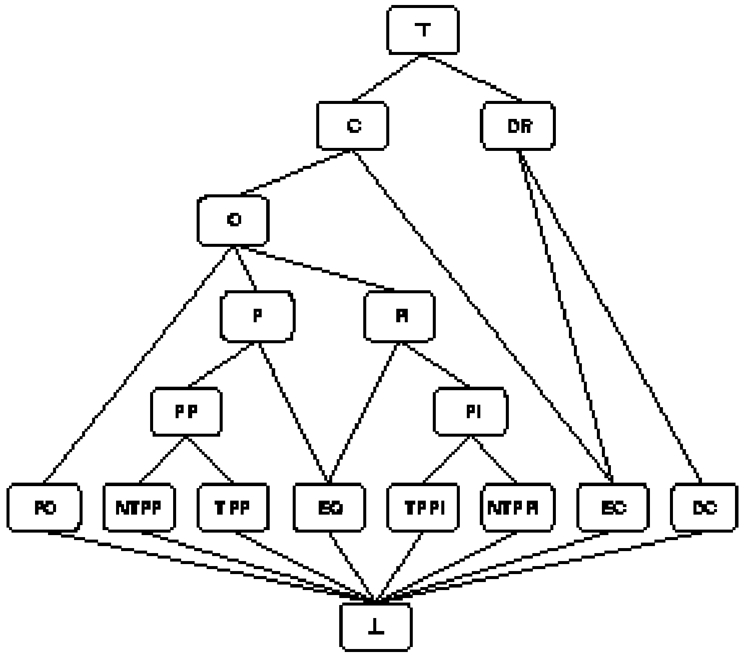
The RCC8-relations





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2. (N)TPP and (N)TPPi are asymmetric
   e.g., TPP xy ⇒ ¬TPPi xy
3. Prove the following theorems
   - PO xy ⇒ O xy
   - ¬(PO xy & NTPP xy)
   - DC xy ⇒ DR xy
   - EC xy ⇒ DR xy
   - ¬(EC xy & DC xy)

   using their definitions and C1-C3