# Qualitative spatio-temporal representation and reasoning: a computational perspective

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"Although there has always been a temptation in KR to set the sights either too low (and provide only a data structuring facility with little or no inference) or too high (and provide a full theorem proving facility), this paper argues for the rich world of representation that lies between these two extremes."

Levesque and Brachman (1985)

## 1 Introduction

Time and space belong to those few fundamental concepts that always puzzled scholars from almost all scientific disciplines, gave endless themes to science fiction writers, and were of vital concern to our everyday life and commonsense reasoning. So whatever approach to AI one takes [Russell and Norvig, 1995], temporal and spatial representation and reasoning will always be among its most important ingredients (cf. [Hayes, 1985]).

Knowledge representation (KR) has been quite successful in dealing *separately* with both time and space. The spectrum of formalisms in use ranges from relatively simple temporal and spatial databases, in which data are indexed by temporal and/or spatial parameters (see e.g. [Srefik, 1995; Worboys, 1995]), to much more sophisticated numerical methods developed in computational geometry (see e.g. [Preparata and Shamos, 1985]) and various qualitative logical theories (see [Stock, 1997; Casati and Varzi, 1999; Cohn and Hazarika, 2001] and references therein). However, despite the modern view of space and time as *space-time* (not only in physics, but in AI as well[1]), apart from approaches

[1] "Events happen in time, but also in space—they have a where as well as a when. They are four-dimensional spatiotemporal entities" [Hayes, 1985]. "The spatial data models currently used as the foundation for geographical information systems (GISs) fall short of conveying the rich and complex ways in which phenomena change over space and time. One of the major limitations of today's systems, for example, is that they capture only a *snapshot* of reality, reliant as they are on databases that contain only current data" [Hornsby and Egenhofer, 2000].

based on classical quantitative models of kinematics (see e.g. [Rajagopalan and Kuipers, 1994; Hays, 1989]), surprisingly little has been done to design *qualitative* spatio-temporal representation formalisms [Vieu, 1991; Galton, 1997; Muller, 1998; Hornsby and Egenhofer, 2000; Wolter and Zakharyaschev, 2000b], let alone implementations. More refs? Or fewer?

Although a deep ontological analysis of qualitative spatio-temporal entities seems still missing [Vieu, 1997], yet there is a quite simple 'naïve' approach to constructing such formalisms. Just take your favorite temporal logic $T$ and your favorite spatial logic $S$, and merge them into a single spatio-temporal hybrid, allowing the desirable amount of interaction between space and time. The construction can be driven either by syntactical or by semantical considerations. In the former case, one joins the axioms of $T$ and $S$ together with some interacting principles (cf. [Muller, 1998]). The next step would be to supply the resulting system with an intended interpretation—to demonstrate which aspects of our intuitive views on space are captured by the theory—and show that they match (i.e., prove soundness and completeness). The example of $\mathsf{RCC}$ [Randell *et al.*, 1992] (as well as general results on multi-dimensional logics [Gabbay *et al.*, 2001]) shows, however, that this can be a hard mathematical problem (see [Gotts, 1996a; Stell, 2000]).

By taking the semantical way—which will be done in this paper—we first integrate the intended models of $T$ and $S$ into a multi-dimensional spatio-temporal structure (as e.g. in Fig. 1), and then combine their languages into a 'super-language' which is capable of speaking about these structures (see e.g. [Wolter and Zakharyaschev, 2000b]). It may be very difficult (if at all possible) to write down axioms for such a system, but for most KR purposes this should not be an obstacle, provided that the interpretation is transparent and convincing, and the system can be supplied with a reasoning procedure.

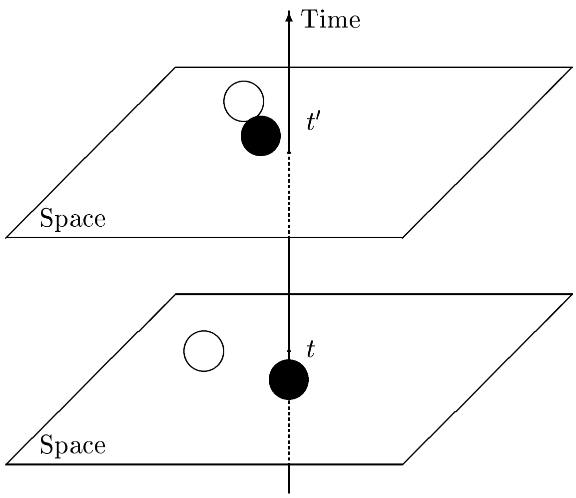


Figure 1: Spatial regions moving in time.

Our intended models of space are a variant of *mereotopological models*: the primitive entities—*regions*—are interpreted as regular closed sets of topological spaces [Grzegorczyk, 1960; Gotts, 1996b] so that any two regions can stand Who else?

in precisely one of the eight relations depicted in Fig. 2 [Egenhofer and Franzosa, 1991; Randell *et al.*, 1992]. As concerns time, we consider three fundamental paradigms: linear point-based time (discrete, dense, etc.), branching point-based time, and linear interval-based time (see e.g. [Allen, 1983; Gabbay *et al.*, 1994; van Benthem, 1996; Gabbay *et al.*, 2000]). The spatial dimension (topological space) is supposed to be always the same, however regions can change their positions with time passing by (see Fig. 1). Thus, our spatio-temporal interpretations can be regarded as the Cartesian products of spatial and temporal structures.

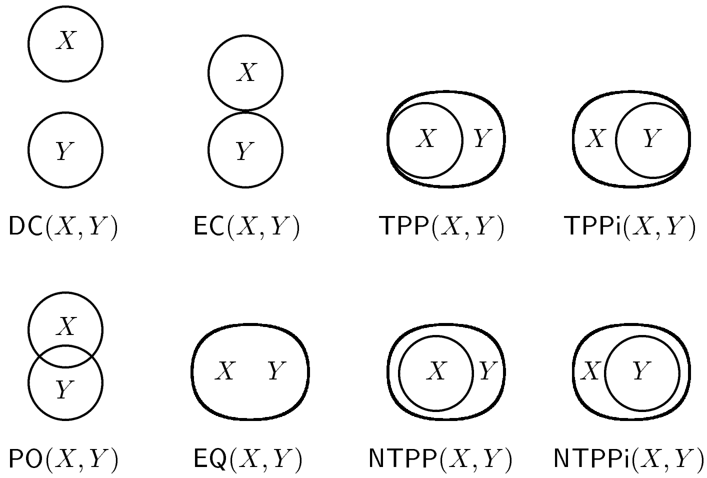


Figure 2: The eight relations between regions.

Having fixed the intended spatio-temporal structures, we still have a rich choice of spatial and temporal languages, in which we can speak about these structures, and a variety of ways to combine the languages. Here we come to the main issue of this paper: to investigate computational properties of spatio-temporal logics. Our concern is to find out which constructors of the languages and which kinds of interaction between them cause a 'bad computational behavior' and which result in 'implementable' spatio-temporal formalisms.

A very important point here is that in this *multi-dimensional* case the 'fundamental tradeoff' is not only between rich first- or higher-order theories on the one hand, and their less expressive (e.g., propositional) fragments on the other, say, between the full region connection calculus RCC [Randell *et al.*, 1992], which is undecidable [Gotts, 1996b; Dornheim, 1998], and its propositional fragment RCC-8 which is decidable [Bennett, 1994] (in fact, NP-complete [Renz and Nebel, 1999]). An interaction between dimensions or, at the syntactical level, between connectives of the spatial and temporal languages can dramatically 'spoil' nice computational properties of the components. The following simple example can serve as a good illustration.

**Example 1.** Consider the *compass logic* of [Venema, 1990] which can be viewed as a sort of 'orientation logic' on the plane. The intended model is the 'map' $\mathbb{N} \times \mathbb{N}$ with the standard orientation; see Fig. 3 (in fact, we can take any infinite grid, say, $\mathbb{R} \times \mathbb{R}$). There are two compass operators $\Diamond_N$ and $\Diamond_E$ on the

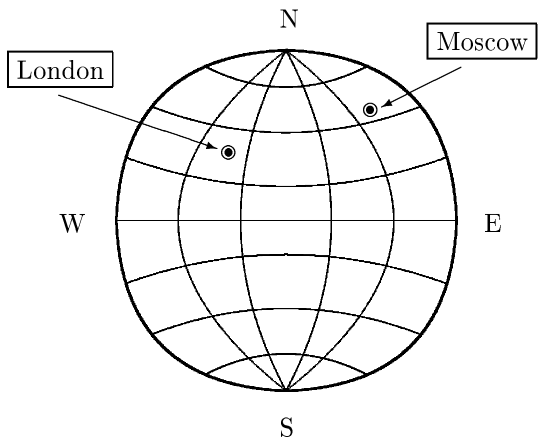


Figure 3: Compass relations.

map, which are interpreted as 'somewhere to the North' and 'somewhere to the East,' respectively (of course, one can add their converses 'somewhere to the South' and 'somewhere to the West' as well), plus we can use the standard Boolean connectives. That Moscow is located to the North-East of London can be expressed in the compass logic by the formula

$$London \to \Diamond_N \Diamond_E Moscow,$$

where *London* and *Moscow* are treated as propositional variables that are either true or false at every point of the map.

This two-dimensional logic can be regarded as a natural combination of two one-dimensional 'compass logics' interpreted on straight lines. The interaction between the dimensions is reflected by the formulas

$$\Diamond_N \Diamond_E \varphi \leftrightarrow \Diamond_E \Diamond_N \varphi,$$
$$\Diamond_E \Box_N \varphi \to \Box_N \Diamond_E \varphi,$$

where $\Box_N$ stands for 'everywhere to the North. The meaning of these formulas is explained by the diagrams in Fig. 4 which say: if there are two black arrows on the map then there are two dashed arrows as well.

Now, the satisfiability problem for the 1D logics is known to be decidable in NP [Ono and Nakamura, 1980], while the satisfiability problem for the 2D compass logic on $\mathbb{N} \times \mathbb{N}$ or $\mathbb{R} \times \mathbb{R}$ is not even recursively enumerable [Spaan, 1993; Marx and Reynolds, 1999; Reynolds and Zakharyaschev, 2001].

Where is the border line between an 'acceptable' and 'unacceptable' computational behavior of KR formalisms? Obviously, the compass logic above is not acceptable from the computational point of view: no algorithm is capable of even enumerating the formulas satisfiable on the map. On the other hand, its one-dimensional fragments are often also regarded as non-tractable [Garey and Johnson, 1979] in view of their NP-hardness. Yet, there is an evidence of Horrocks [1998] who demonstrated that "some of the very expressive description logics for which tableaux algorithms are now available may also be usable

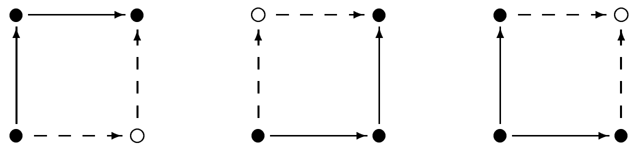

Figure 4: Commutativity and Church–Rosser properties.

in realistic applications." Here and in [Horrocks *et al.*, 1999] 'very expressive' means PSPACE-hard and EXPTIME-hard, respectively. Hustadt and Schmidt [2000] successfully used a full first-order prover for dealing with EXPTIME-complete modal logics. And MONA (see [Klarlund *et al.*, 2000] and references therein) is a good example of an implementation of decision procedures for theories with a non-elementary worst-case complexity.[2] A possible explanation of this phenomenon is that "in all practically occurring situations the worst case never seems to happen. The reason is that definitions occurring in practice are somehow well-structured. ... and this does not only hold for knowledge representation systems based on description logic but also for object-oriented database systems" [Nebel, 1996].

Of course, only experiments can show whether this or that KR formalism can be applied in practice. In this paper, however, we suggest to qualify a logic as having an acceptable computational behavior if it is

- decidable and
- supported by a potentially implementable decision algorithm.

The organization of the paper is very simple. In Sections 2 and 3 we introduce the spatial and temporal components of the spatio-temporal logics to be constructed in Section 4. We define both the syntax and the intended semantics of the logics, illustrate their expressive power by multiple examples, and focus attention on their computational behavior. It is to be noted from the very beginning that we are not putting forward a novel spatio-temporal paradigm. Nor are we designing *the* spatio-temporal KR formalism suitable for all potential applications in GISs, computer vision, robotics, image retrieval, etc. Our aim is more modest: we combine (some of) the existing spatial and temporal logics and analyze the computational behavior of the resulting hierarchy of spatio-temporal hybrids. As the field of spatio-temporal representation and reasoning is still at the 'embryo' stage, the paper contains a considerable number of open problems.

Although all technical proofs are omitted, we nevertheless try to give the reader the underlying ideas in the hope of sharing our excitement about this interesting and promising field of KR based on multiple connections to geometry, algebra, topology, modal and temporal logics, and other disciplines.

[2] "Perhaps surprisingly, this complexity also contributes to successful applications, since it is provably linked to the succinctness of the logics" [Klarlund *et al.*, 2000].

# 2 Of space

There are different approaches to qualitative spatial representation in AI (comprehensive surveys can be found in [Vieu, 1997; Casati and Varzi, 1999; Cohn and Hazarika, 2001]). Here we consider only one, perhaps the most influential of them, which takes extended regions of space as the primitive spatial entity. Properties of regions are usually defined by first-order axiomatic theories (see e.g. [Clarke, 1981; Randell *et al.*, 1992]) the (explicit or implicit) intended models of which are topological, in particular, Euclidean spaces.

## 2.1 Topological spaces

**Definition 2 (topological space).** A *topological space* is a pair $\mathfrak{T} = \langle U, \mathbb{I} \rangle$ in which $U$ is a non-empty set, the *universe* of the space, and $\mathbb{I}$ is the *interior operator* on $U$ satisfying the following *Kuratowski axioms*: for all $X, Y \subseteq U$,

$$\mathbb{I}(X \cap Y) = \mathbb{I}X \cap \mathbb{I}Y, \quad \mathbb{I}X \subseteq \mathbb{I}\mathbb{I}X, \quad \mathbb{I}X \subseteq X, \quad \mathbb{I}U = U.$$

The operator dual to $\mathbb{I}$ is called the *closure operator* and denoted by $\mathbb{C}$; thus, for every $X \subseteq U$, $\mathbb{C}X = U - \mathbb{I}(U - X)$ (or $\mathbb{C}X = -\mathbb{I} - X$, for short). A set $X \subseteq U$ is called *open* if $X = \mathbb{I}X$ ($\mathbb{I}X$ is known as the *interior* of $X$) and *closed* if $X = \mathbb{C}X$ ($\mathbb{C}X$ is the *closure* of $X$). The set $\mathbb{C}X - \mathbb{I}X$ is called the *boundary* of $X$.

In this paper, we will need only two kinds of topological spaces.

**Example 3 (Euclidean spaces).** Let $X$ be a set of real numbers, i.e, $X \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be *interior* in $X$ if there is some $\epsilon > 0$ such that the whole open interval $(x - \epsilon, x + \epsilon)$ belongs to $X$. The interior $\mathbb{I}X$ of $X$ is defined then as the set of all interior points in $X$. It is not hard to check that $\langle \mathbb{R}, \mathbb{I} \rangle$ is a topological space; it is called the *one-dimensional Euclidean space.* Open sets in $\langle \mathbb{R}, \mathbb{I} \rangle$ are (possibly infinite) unions of open intervals $(a, b)$, where $a \leq b$. The closure of $(a, b)$, for $a < b$, is the closed interval $[a, b]$, with the end points $a$ and $b$ being its boundary. In the same manner one can define *higher-dimensional Euclidean spaces* based the universes $\mathbb{R}^n$ for $n > 1$ (in the definition of interior points $x$ one should take $n$-dimensional $\epsilon$-neighborhoods of $x$).

There can be different views on what sets of a topological space $\mathfrak{T} = \langle U, \mathbb{I} \rangle$ can be taken as interpretations of spatial regions (for a discussion consult [Vieu, 1997; Gotts, 1996a]).[3] Following [Gotts, 1996a], we interpret regions only as *regular closed* sets, i.e., sets $X$ such that $X = \mathbb{C}\mathbb{I}X$ (an alternative would be to take *regular open* sets $X$ for which $X = \mathbb{I}\mathbb{C}X$). For example, the circle

Who was the first?

$$C(a, r) = \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} \leq r\}$$

with center $a = (a_1, a_2)$ and of radius $r > 0$ on the Euclidean plane is a regular closed set, while the balloon obtained by attaching to $C(a, r)$ a thread (e.g. a segment of a straight line) is not regular closed, because the interior of the thread in $\mathbb{R}^2$ is empty.

[3] Actually, the choice is determined by the way of characterizing a relation of connection between two regions. Our interpretation reflects the following definition: regions $X$ and $Y$ are connected iff $\mathbb{C}X \cap \mathbb{C}Y \neq \emptyset$. For a comparison of different definitions consult [Cohn and Varzi, 1998].

An important property of topological spaces is that any, even infinite, union (intersection) of open (closed) sets is open (respectively, closed). An infinite union (intersection) of closed (open) sets is not necessarily closed (open). For example, in $\mathbb{R}$ we have:

$$\bigcup_{n=1}^{\infty}[1/n, 1-1/n] = (0,1), \qquad \bigcap_{n=1}^{\infty}[-1/n, 1/n] = \{0\}. \tag{1}$$

**Example 4 (Kripke spaces).** Recall that a *quasi-order* is a pair $\mathfrak{G} = \langle V, S\rangle$, where $S$ is a reflexive and transitive binary relation on $V \neq \emptyset$. With every quasi-order $\mathfrak{G}$ one can associate a topological space $\mathfrak{T}_{\mathfrak{G}} = \langle V, \mathbb{I}_{\mathfrak{G}}\rangle$ by taking, for every $X \subseteq V$,

$$\mathbb{I}_{\mathfrak{G}}X = \{x \in X : \forall y \in V \ (xSy \to y \in X)\}. \tag{2}$$

We call $\mathfrak{T}_{\mathfrak{G}}$ the *Kripke space determined by* $\mathfrak{G}$ (the reason for this name will be explained in Section 2.2.4; these spaces are also known as Alexandrov spaces). The closure operator $\mathbb{C}_{\mathfrak{G}}$ on $\mathfrak{T}_{\mathfrak{G}}$ is defined then by

$$\mathbb{C}_{\mathfrak{G}}X = \{y \in V : \exists x \in X \ ySx\}. \tag{3}$$

It follows from (2) and (3) that arbitrary unions (intersections) of closed (open) sets of Kripke spaces are closed (respectively, open).

It is not hard to see that, for any two regions $X$ and $Y$ in a topological space $\mathfrak{T} = \langle U, \mathbb{I}\rangle$, one and only one of the following eight relations can hold between $X$ and $Y$ (see Fig. 2):

$$\begin{array}{lll}
\mathsf{DC}(X,Y) & \text{iff} & \neg\exists x \ x \in X \cap Y,\\
\mathsf{EQ}(X,Y) & \text{iff} & \forall x \ (x \in X \leftrightarrow x \in Y),\\
\mathsf{PO}(X,Y) & \text{iff} & \exists x \ (x \in \mathbb{I}X \cap \mathbb{I}Y) \land \exists x \ (x \in \mathbb{I}X \cap -Y) \land \exists x \ (x \in -X \cap \mathbb{I}Y),\\
\mathsf{EC}(X,Y) & \text{iff} & \exists x \ (x \in X \cap Y) \land \neg\exists x \ (x \in \mathbb{I}X \cap \mathbb{I}Y),\\
\mathsf{TPP}(X,Y) & \text{iff} & \forall x \ (x \in -X \cup Y) \land \exists x \ (x \in X \cap -\mathbb{I}Y) \land \exists x \ (x \in -X \cap Y),\\
\mathsf{TPPi}(X,Y) & \text{iff} & \mathsf{TPP}(Y,X),\\
\mathsf{NTPP}(X,Y) & \text{iff} & \forall x \ (x \in -X \cup \mathbb{I}Y) \land \exists x \ (x \in -X \cap Y),\\
\mathsf{NTPPi}(X,Y) & \text{iff} & \mathsf{NTPP}(Y,X).
\end{array}$$

In English, these relations can be described as Disconnection, Equality, Partial Overlap, External Connection, Tangential Proper Part, Non-Tangential Proper Part, and the inverses of the last two.

In view of this property of being *jointly exhaustive and pairwise disjoint*, the eight relations above play a fundamental role in spatial representation and reasoning (the same as Allen's 13 relations between time intervals; see Section 3.4). We will call them the *basic relations* or the RCC-8 *relations* (or *predicates*).

## 2.2 Spatial logics

Languages of different expressive power can be used to talk about regions in topological spaces.

| | | |
|---|---|---|
| $\mathsf{DC}(X,Y)$ | $=$ | $\neg\mathsf{C}(X,Y)$ |
| $\mathsf{P}(X,Y)$ | $=$ | $\forall Z\ (\mathsf{C}(Z,X) \to \mathsf{C}(Z,Y))$ |
| $\mathsf{EQ}(X,Y)$ | $=$ | $\mathsf{P}(X,Y) \land \mathsf{P}(Y,X)$ |
| $\mathsf{O}(X,Y)$ | $=$ | $\exists Z\ (\mathsf{P}(Z,X) \land \mathsf{P}(Z,Y))$ |
| $\mathsf{PO}(X,Y)$ | $=$ | $\mathsf{O}(X,Y) \land \neg\mathsf{P}(X,Y) \land \neg\mathsf{P}(Y,X)$ |
| $\mathsf{EC}(X,Y)$ | $=$ | $\mathsf{C}(X,Y) \land \neg\mathsf{O}(X,Y)$ |
| $\mathsf{PP}(X,Y)$ | $=$ | $\mathsf{P}(X,Y) \land \neg\mathsf{P}(Y,X)$ |
| $\mathsf{TPP}(X,Y)$ | $=$ | $\mathsf{PP}(X,Y) \land \exists Z\ (\mathsf{EC}(Z,X) \land \mathsf{EC}(Z,Y))$ |
| $\mathsf{NTPP}(X,Y)$ | $=$ | $\mathsf{PP}(X,Y) \land \neg\exists Z\ (\mathsf{EC}(Z,X) \land \mathsf{EC}(Z,Y))$ |

Table 1: Some relations between spatial regions, defined in terms of $\mathsf{C}$.

### 2.2.1 First-order logics

Often, logical formalisms for qualitative spatial representation are formulated as first-order theories [Whitehead, 1929; Clarke, 1981; Randell *et al.*, 1992; Casati and Varzi, 1999]. For instance, the language of $\mathsf{RCC}$ consists of individual variables $X, Y, \dots$ (understood as variables over regions), the individual constant $\mathsf{U}$ (for the universal region), the binary predicate $\mathsf{C}(X,Y)$ (read as '$X$ connects with $Y$'), a number of functions such as $\mathsf{sum}(X,Y)$, $\mathsf{compl}(X)$, $\mathsf{prod}(X,Y)$, the Boolean logical connectives, and the quantifiers $\forall$ and $\exists$. The eight basic predicates are defined via $\mathsf{C}$ as in Table 1 (where $\mathsf{P}$ stands for 'part,' $\mathsf{O}$ for 'overlaps,' and $\mathsf{PP}$ for 'proper part'), and the axioms of $\mathsf{RCC}$ include, in particular,

$$\begin{aligned}
&\forall X\ \mathsf{C}(X,X),\\
&\forall X,Y\ (\mathsf{C}(X,Y) \to \mathsf{C}(Y,X)),\\
&\forall X\ \mathsf{C}(X,\mathsf{U}),\\
&\forall X \exists Y\ \mathsf{NTPP}(Y,X),\\
&\forall X,Y,Z\ \big(\mathsf{C}(Z,\mathsf{sum}(X,Y)) \leftrightarrow \mathsf{C}(Z,X) \lor \mathsf{C}(Z,Y)\big).
\end{aligned}$$

Unfortunately, from the computational point of view, full $\mathsf{RCC}$ turns out to be too expressive: as was shown in [Gotts, 1996b; Dornheim, 1998] (and actually follows from [Grzegorczyk, 1951]), it is undecidable. Another problem with $\mathsf{RCC}$ is its semantical characterization. For example, it is an open question whether $\mathsf{RCC}$ is sound and complete with respect to topological interpretations. Is it so?

Of course, one can change direction and start from semantics. If we are satisfied with the mereotopological model for qualitative spatial representation, then we can use as a variant of spatial logic the set of all first-order formulas in a proper signature, say, containing the eight basic predicates in Fig. 2, that hold in all topological models defined as follows (cf. [Dornheim, 1998]):

**Definition 5 (topological model).** A *topological model* is a structure of the form

$$\mathfrak{S} = \left\langle \mathcal{R}(\mathfrak{T}); \mathsf{DC}^{\mathfrak{T}}, \mathsf{EQ}^{\mathfrak{T}}, \mathsf{PO}^{\mathfrak{T}}, \mathsf{EC}^{\mathfrak{T}}, \mathsf{TPP}^{\mathfrak{T}}, \mathsf{TPPi}^{\mathfrak{T}}, \mathsf{NTPP}^{\mathfrak{T}}, \mathsf{NTPPi}^{\mathfrak{T}} \right\rangle, \tag{4}$$

where $\mathfrak{T}$ is a topological space, $\mathcal{R}(\mathfrak{T})$ the set of all regular closed subsets in $\mathfrak{T}$, and $\mathsf{DC}^{\mathfrak{T}}$, $\mathsf{EQ}^{\mathfrak{T}}$, $\mathsf{PO}^{\mathfrak{T}}$, $\mathsf{EC}^{\mathfrak{T}}$, $\mathsf{TPP}^{\mathfrak{T}}$, $\mathsf{TPPi}^{\mathfrak{T}}$, $\mathsf{NTPP}^{\mathfrak{T}}$, $\mathsf{NTPPi}^{\mathfrak{T}}$ are the basic predicates on $\mathcal{R}(\mathfrak{T})$ defined as above.

Let $\mathfrak{a}$ be an *assignment* in $\mathfrak{S}$ associating with every region variable $X$ a set $\mathfrak{a}(X)$ in $\mathcal{R}(\mathfrak{T})$. A first-order formula $\varphi(X_1, \dots, X_n)$ in the signature of the RCC-8 predicates and with free variables $X_1, \dots, X_n$ is *satisfied* in $\mathfrak{S}$ under $\mathfrak{a}$ ($\mathfrak{S} \models^{\mathfrak{a}} \varphi$ in symbols) if $\mathfrak{S} \models \varphi[\mathfrak{a}(X_1), \dots, \mathfrak{a}(X_n)]$ in the standard model-theoretic sense.

Unfortunately, even this simplified approach turns out to be computationally unacceptable: as follows from [Grzegorczyk, 1951], this logic is undecidable as well. (Booleans expressible via C.)

Check!!. What about axiomatizability?

### 2.2.2 RCC-8

As the eight basic region-relations play so important role in spatial representation and reasoning [Egenhofer and Franzosa, 1991; Egenhofer, 1991; Smith and Park, 1992], to obtain a computationally well-behaved spatial formalism, we can sacrifice quantification and consider the quantifier-free fragment of the logic of topological models defined above. This fragment is known as RCC-8. Thus, RCC-8 *formulas* are simply Boolean combinations of the RCC-8 predicates.

**Definition 6 (consequence).** Say that an RCC-8 formula $\varphi$ is a *consequence* of a set $\Sigma$ of RCC-8 formulas if for every topological model $\mathfrak{S}$ and every assignment $\mathfrak{a}$ in it, we have $\mathfrak{S} \models^{\mathfrak{a}} \varphi$ whenever $\mathfrak{S} \models^{\mathfrak{a}} \Sigma$. In this case we write $\Sigma \models \varphi$.

For example, using the language of RCC-8 we can compose spatial knowledge bases like

$$\begin{aligned}
&\mathsf{EC}(\textit{Catalunya, France}),\\
&\mathsf{TPP}(\textit{Catalunya, Spain}) \vee \mathsf{NTPP}(\textit{Catalunya, Spain}),\\
&\mathsf{DC}(\textit{Spain, France}) \vee \mathsf{EC}(\textit{Spain, France}),\\
&\mathsf{NTPP}(\textit{Paris, France}).
\end{aligned}$$

The formulas $\mathsf{EC}(\textit{Spain, France})$, $\mathsf{TPP}(\textit{Catalunya, Spain})$, and $\mathsf{DC}(\textit{Spain, Paris})$ are then consequences of this knowledge base.

It should be clear that $\Sigma \models \varphi$ holds iff the formula $\neg\varphi \wedge \bigwedge \Sigma$ is not satisfiable in topological models. Thus, to understand the computational properties of RCC-8, we can confine ourselves to considering only the *satisfiability problem*.

That this problem is decidable was first observed by Bennett [1994] who encoded RCC-8 into a decidable propositional modal logic. And later Renz and Nebel [1999] showed that the satisfiability problem for RCC-8 formulas is NP-complete. For more details see Section 2.2.4.

### 2.2.3 BRCC-8

One apparent 'deficit' of RCC-8 is that it operates only with *atomic* regions. We can't form unions ($\sqcup$) or intersections ($\sqcap$) of regions to say, for instance, that

$$\mathsf{EQ}(\textit{EU}, \textit{Spain} \sqcup \textit{Italy} \sqcup \dots)$$

('the EU consists of Spain, Italy, etc.'),

$$\mathsf{P}(Alps, Italy \sqcup France \sqcup \dots)$$

('the Alps are located in Italy, France, etc.'),

$$\mathsf{EC}(Austria, Alps \sqcap Italy)$$

('Austria is externally connected to the alpine part of Italy'), and deduce from these that if $\mathsf{EC}(X, EU)$, for some country $X$, then $\mathsf{EC}(X, Y)$ for some country $Y$ in the EU, or that there is a country $Z$ such that $\mathsf{TPP}(Z, EU)$ (i.e., '$Z$ is a tangential proper part of the EU'). Note by the way that the last formula is a correct conclusion only if we interpret our formulas in Euclidean (or, more generally, connected[4]) topological spaces (and if there are non-EU countries): in a discrete topological space (where all sets are open) the EU may be an open set with empty boundary. This simple observation and the result of [Renz, 1998], according to which every satisfiable RCC-8 formula is satisfiable in all Euclidean spaces $\mathbb{R}^n$, $n \geq 1$, show that the Boolean operations on region terms indeed increase the expressive power of RCC-8.

**Definition 7 (boolean region term).** A *Boolean region term* is just a combination of region variables using the Boolean operators $\sqcup$, $\sqcap$, and $\neg$.

Denote by BRCC-8 the extension of RCC-8 which allows the use of Boolean region terms as arguments of the RCC-8 predicates. As the Boolean operators do not in general preserve the property of being regular closed, we have to adjust the interpretation of Boolean region terms in a topological model $\mathfrak{S}$ of the form (4) by taking, for region terms $t$ and $t'$,

$$\begin{aligned} \mathfrak{a}(t \sqcup t') &= \mathbb{CI}\,(\mathfrak{a}(t) \cup \mathfrak{a}(t')) = \mathfrak{a}(t) \cup \mathfrak{a}(t'), \\ \mathfrak{a}(t \sqcap t') &= \mathbb{CI}\,(\mathfrak{a}(t) \cap \mathfrak{a}(t')), \\ \mathfrak{a}(\neg t) &= \mathbb{CI}\,(U - \mathfrak{a}(t)). \end{aligned}$$

Thus, every region term is interpreted as a regular closed set of $\mathfrak{T}$. Note that $\mathfrak{a}(X \sqcap \neg X) = \emptyset$ and $\mathfrak{a}(X \sqcup \neg X) = U$ for any $\mathfrak{a}$ and $\mathfrak{T}$. We denote the region terms $X \sqcap \neg X$ and $X \sqcup \neg X$ by $\bot$ and $\top$, respectively. The constraint $\neg\mathsf{EQ}(X, \bot)$ asserts that $X$ is a non-empty region.

The computational behavior of BCCR-8 in arbitrary topological models is precisely the same as that of RCC-8. However, if only Euclidean topological models are regarded as possible interpretations, the satisfiability problem for BCCR-8 formulas becomes PSPACE-complete (for details consult [Wolter and Zakharyaschev, 2000a]).

### 2.2.4 Modal logics as spatial logics

The proof of the decidability of RCC-8 in [Bennett, 1994] brought in sight another kind of formalism which can be used as a spatial logic. In fact, the logic was introduced independently by Orlov [1928], Lewis in [?], and Gödel [1933] without any intention to reason about space. Lewis baptized the logic as S4

[4] A topological space is called *connected* if it can't be represented as a union of two disjoint open sets.

and understood it as a logic of necessity and possibility, that is as a *modal logic*. Besides the Boolean connectives and propositional variables, its language contains two modal operators $\Box$ ('it is necessary;' Orlov and Gödel treated $\Box$ as 'it is provable') and $\Diamond$ ('it is possible'). The axiom shemata of S4 are those of classical propositional calculus, three modal schemata:

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi), \quad \Box\varphi \to \varphi, \quad \Box\varphi \to \Box\Box\varphi,$$

and two inference rules: modus ponens and necessitation $\varphi/\Box\varphi$. The possibility operator is defined as dual to $\Box$, i.e., $\Diamond\varphi = \neg\Box\neg\varphi$.

In the late thirties and early forties several logicians [Stone, 1937; Tarski, 1938; Tsao-Chen, 1938; McKinsey, 1941] noticed that S4 can be interpreted in topological spaces. Actually, there is a striking similarity between the axioms of S4 and Kuratowski's axioms for the interior operator. (The first schema and rule of necessitation can be replaced with $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$ and $\Box\top$, corresponding to the first and the last topological axioms.)

Suppose that an assignment $\mathfrak{v}$ in a topological space $\mathfrak{T} = \langle U, \mathbb{I}\rangle$ is a map from the set of propositional variables in S4 to $2^U$. We then inductively extend $\mathfrak{v}$ to all modal formulas by interpreting $\Box$ as $\mathbb{I}$, $\Diamond$ as $\mathbb{C}$, $\wedge$ as $\cap$, and $\neg$ as $-$. Now we say that a modal formula $\varphi$ is *satisfied* in $\mathfrak{T}$ under $\mathfrak{v}$ if $\mathfrak{v}(\varphi) \neq \emptyset$; $\varphi$ is *valid in* $\mathfrak{T}$ ($\mathfrak{T} \models \varphi$, in symbols) if $\mathfrak{v}(\varphi) = U$. It turns out that S4 is sound and complete with respect to this interpretation: a modal formula $\varphi$ is derivable in S4 iff $\varphi$ is valid in all topological spaces iff $\varphi$ is valid in any $n$-dimensional Euclidean space ($n \geq 1$) [McKinsey, 1941; McKinsey and Tarski, 1944]. A remarkable result due to Dummett and Lemmon [1959] and Kripke [1963] is that S4 is complete with respect to *finite* Kripke spaces (Kripke used quasi-orders to define his possible world semantics for S4).[5]

Thus, S4 can be regarded as a 'logic of topological spaces.' We can increase the expressive power of S4 by adding to it one more pair of modal operators $\boxed{\forall}$ and $\langle\exists\rangle$, known as the *universal modalities*. The topological meaning of $\boxed{\forall}$ and $\langle\exists\rangle$ is 'for all points in the space' and 'for some point in the space,' respectively. More precisely, for every formula $\varphi$ in the extended language and every topological space $\mathfrak{T} = \langle U, \mathbb{I}\rangle$ with an assignment $\mathfrak{v}$, we have:

$$\mathfrak{v}(\boxed{\forall}\varphi) = \begin{cases} U & \text{if } \mathfrak{v}(\varphi) = U, \\ \emptyset & \text{otherwise;} \end{cases} \qquad \mathfrak{v}(\langle\exists\rangle\varphi) = \begin{cases} U & \text{if } \mathfrak{v}(\varphi) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

The set of all formulas in this language that are valid in all topological spaces is denoted by $\mathsf{S4}_u$; it can be axiomatized by adding to S4 the schemata

$$\Diamond\varphi \to \Box\Diamond\varphi \quad \text{and} \quad \boxed{\forall}\varphi \to \Box\varphi.$$

According to [Goranko and Passy, 1992], $\mathsf{S4}_u$ is also complete with respect to finite Kripke spaces. Note, however, that in constrast to S4 itself, $\mathsf{S4}_u$ is not complete with respect to Euclidean spaces. The set of formulas valid in all Euclidean spaces is strictly larger than $\mathsf{S4}_u$. It was axiomatized in [Shehtman, 1999] by adding to $\mathsf{S4}_u$ the schemata $\boxed{\forall}(\Box\varphi \vee \Box\neg\varphi) \to \boxed{\forall}\varphi \vee \boxed{\forall}\neg\varphi$.

[5] This story is really amazing. In 1908, Brouwer introduced intuitionistic logic Int; later he became also famous in topology. Orlov and Gödel defined S4 in order to interpret intuitionistic logic in classical one. Open sets in a topological space form a complete Heyting algebra, which is a model of Int, and can be used as a model of RCC [Stell and Worboys, 1979; Stell, 2000].

$\mathsf{S4}_u$ is expressive enough to encode the topological meaning of the $\mathsf{RCC}$-8 predicates and that of Boolean region terms.[6] Indeed, let us denote the box and the diamond of $\mathsf{S4}$ by, respectively, $\boldsymbol{I}$ and $\boldsymbol{C}$ (to emphasize their topological interpretation as the interior and closure operators). For a Boolean region term $t$, define inductively a modal formula $t^*$ by taking:

$$
\begin{aligned}
X_i^* &= \boldsymbol{CI}p_i, \ \ (X_i \text{ is a region variable, } p_i \text{ a propositional variable}),\\
(t_1 \sqcap t_2)^* &= \boldsymbol{CI}(t_1^* \wedge t_2^*),\\
(t_1 \sqcap t_2)^* &= \boldsymbol{CI}(t_1^* \vee t_2^*),\\
(\neg t)^* &= \boldsymbol{CI}\neg t^*.
\end{aligned}
$$

Then, with every atomic $\mathsf{BRCC}$-8 formula $P(s,t)$ we associate a modal formula $(P(s,t))^*$ defined by:

$$
\begin{aligned}
(\mathsf{DC}(s,t))^* &= \neg \Diamond_{\exists}(s^* \wedge t^*),\\
(\mathsf{EQ}(s,t))^* &= \Box_{\forall}(s^* \leftrightarrow t^*),\\
(\mathsf{PO}(s,t))^* &= \Diamond_{\exists}(\boldsymbol{I}s^* \wedge \boldsymbol{I}t^*) \wedge \Diamond_{\exists}(\boldsymbol{I}s^* \wedge \neg t^*) \wedge \Diamond_{\exists}(\neg s^* \wedge \boldsymbol{I}t^*),\\
(\mathsf{EC}(s,t))^* &= \Diamond_{\exists}(s^* \wedge t^*) \wedge \neg \Diamond_{\exists}(\boldsymbol{I}s^* \wedge \boldsymbol{I}t^*),\\
(\mathsf{TPP}(s,t))^* &= \Box_{\forall}(\neg s^* \vee t^*) \wedge \Diamond_{\exists}(s^* \wedge \boldsymbol{C}\neg t^*) \wedge \Diamond_{\exists}(\neg s^* \wedge t^*),\\
(\mathsf{NTPP}(s,t))^* &= \Box_{\forall}(\neg s^* \vee \boldsymbol{I}t^*) \wedge \Diamond_{\exists}(\neg s^* \wedge t^*).
\end{aligned}
$$

Finally, given a $\mathsf{BRCC}$-8 formula $\varphi$, denote by $\varphi^*$ the result of replacing all occurrences of atomic formulas $P(s,t)$ in $\varphi$ by $(P(s,t))^*$.

Since the definition of the translation $\cdot^*$ mimics the definition of the $\mathsf{RCC}$-8 predicates and since the formula $\boldsymbol{CICI}\varphi \leftrightarrow \boldsymbol{CI}\varphi$ is provable in $\mathsf{S4}$, we immediately obtain the following theorem the original $\mathsf{RCC}$-8 version of which is due to [Bennett, 1994] (see also [Wolter and Zakharyaschev, 2000a]):

**Theorem 8.** *For every* $\mathsf{BRCC}$-8 *formula* $\varphi$*, the following conditions are equivalent:*
(i) $\varphi$ *is satisfiable in a topological model;*
(ii) $\varphi^*$ *is satisfiable in a topological space;*
(iii) $\varphi^*$ *is satisfiable in a finite Kripke space.*

As a consequence we have:

**Corollary 9.** *The satisfiability problem for* $\mathsf{BRCC}$-8 *formulas is decidable.*

The modal translation $\varphi^*$ of a $\mathsf{BRCC}$-8 formula $\varphi$ has a rather special form. Renz [1998] used this form to show that satisfiable $\mathsf{RCC}$-8 formulas can be satisfied in very simple Kripke spaces, namely in those determined by quasi-orders we call quasisaws.

A *quasisaw* is a partial order $\mathfrak{G} = \langle W, R \rangle$ every point in which has at most two successors, with these successors being $R$-incomparable. An example of a quasisaw is shown in Fig. 5. It should be clear that if an $\mathsf{S4}_u$-formula is satisfied in a quasisaw then it is satisfied in a disjoint union of forks (defined in Fig. 5) as well. The following generalization of Renz's result was proved in [Wolter and Zakharyaschev, 2000a].

[6] Recently, the expressive power of the language of $\mathsf{S4}_u$ has been characterized in terms of bisimulations by Aiello and van Benthem [2000]. The associated topo-games have been used in [Aiello, 2001] to measure a difference between spatial regions.

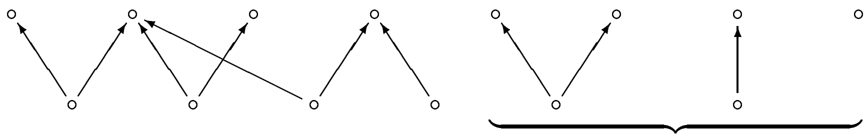


Figure 5: Quasisaw.

**Theorem 10.** *A* BRCC-8 *formula $\varphi$ is satisfiable iff $\varphi^*$ is satisfiable in the Kripke space determined by a quasisaw containing $\leq \ell(\varphi^*)$ forks, where $\ell(\varphi^*)$ is the length of $\varphi^*$.*

Thus, the satisfiability problem for BRCC-8 formulas $\varphi$ in topological models reduces to the satisfiability problem for their modal translations $\varphi^*$ in quasisaws which are disjoint unions of forks. We can make one step further by observing that the latter problem can be reduced to the satisfiability of first-order formulas with a *single* variable. The idea behind this reduction is to represent every subformula $\psi$ of $\varphi^*$ by means of three first-order formulas $\psi^b$, $\psi^l$, $\psi^r$ which encode the 'behavior' of $\psi$ at the three points of a fork. More precisely, we define inductively three translations $\cdot^b$, $\cdot^l$, and $\cdot^r$ by taking

$$
\begin{aligned}
p^i &= P^i(x),\ p \text{ a propositional variable, for } i \in \{b,l,r\},\\
(\psi \circ \chi)^i &= \psi^i \circ \chi^i,\ \text{ for } i \in \{b,l,r\} \text{ and } \circ \in \{\wedge, \vee\},\\
(\neg\psi)^i &= \neg\psi^i,\ \text{ for } i \in \{b,l,r\},\\
(\boldsymbol{I}\psi)^b &= \psi^b \wedge \psi^l \wedge \psi^r,\\
(\boldsymbol{I}\psi)^i &= \psi^i,\ \text{ for } i \in \{l,r\},\\
(\boxed{\forall}\psi)^i &= \forall x\ (\psi^b \wedge \psi^l \wedge \psi^r),\ \text{ for } i \in \{b,r,l\}.
\end{aligned}
$$

Finally, we define the translation $\varphi^\dagger$ of a BRCC-8 formula $\varphi$ into the *one-variable fragment* of first-order logic as $(\varphi^*)^b$. It should be clear that the length of $\varphi^\dagger$ is polynomial in the length of $\varphi$.

**Theorem 11.** *A* BRCC-8 *formula $\varphi$ is satisfiable in a topological model iff $\varphi^\dagger$ is a satisfiable first-order formula.*

As is well-known, the satisfiability problem for first-order formulas with one variable is NP-complete (the one-variable fragment of first-order logic is a notational variant of the propositional modal logic S5). As a consequence, we immediately obtain the following generalization of a result of [Renz and Nebel, 1999]:

**Theorem 12.** *The satisfiability problem for* BRCC-8 *formulas in topological models is NP-complete.*

(Remember that satisfiability of BRCC-8 formulas in connected topological spaces is PSPACE-complete.)

# 3 Of time

Let us now turn to semantical structures representing time and languages designed for speaking about these structures.

**Definition 13 (flow of time).** By a *flow of time* we mean any strict partial order $\mathfrak{F} = \langle W, < \rangle$, where $W$ is a non-empty set of *time points* and $<$ a (transitive and irreflexive) *precedence* relation on $W$.

Depending on applications, one can distinguish between various kinds of flows of time. For example, a linear discrete flow like $\langle \mathbb{N}, < \rangle$ can represent ticks of the computer clock or years AD. A linear dense flow like $\langle \mathbb{Q}, < \rangle$ or $\langle \mathbb{R}, < \rangle$ reflects the continuity of time. A branching flow $\langle W, < \rangle$, where $<$ is a tree order on $W$ (for a precise definition see Section 3.2) suggests that the future is non-deterministic, while the past is determined. For more discussions consult [Gabbay *et al.*, 1994, 2000].

## 3.1 Linear time

As in the case of space, we can choose between different languages to speak about flows of time.

### 3.1.1 First-order logic

First, we can take the first-order language $\mathcal{L}^<$ with one binary predicate $<$, interpreted by the precedence relation of a given flow of time $\mathfrak{F} = \langle W, < \rangle$, an infinite list $P_0, P_1, \dots$ of unary predicates for expressing properties of the time points, and individual variables $x_0, x_1, \dots$ ranging over these points. Formulas of $\mathcal{L}^<$ are built from atoms of the form $P_i(x_j)$ and $x_l < x_m$ by means of the Booleans $\wedge$ and $\neg$, and the first-order quantifiers $\forall x_i$ and $\exists x_i$.

The language $\mathcal{L}^<$ and its relation to automata has been thoroughly investigated [Büchi, 1962; Gurevich, 1964; Läuchli and Leonard, 1966; Stockmeyer, 1974; Meyer, 1975; Burgess and Gurevich, 1985]. In particular, the following results have been obtained:

**Theorem 14.** *The satisfiability problem for $\mathcal{L}^<$-formulas is decidable in the following classes of flows of time: all strict linear orders, $\{\langle \mathbb{R}, < \rangle\}$, $\{\langle \mathbb{Q}, < \rangle\}$, $\{\langle \mathbb{Z}, < \rangle\}$, $\{\langle \mathbb{N}, < \rangle\}$. However, in all these cases, the satisfiability problem is non-elementary.* Is it so?

Thus, reasoning about time with first-order logic is 'very expensive.' On the other hand, in our everyday life we rarely use explicit quantification over time points, preferring expressions like 'tomorrow,' 'always,' 'eventually,' 'since,' etc., which do not mention time points explicitly.

### 3.1.2 Propositional temporal logic

Temporal logic, as opposed to first-order logic, is an approach to reasoning about time (and computation) using such expressions as temporal connectives and not allowing for explicit quantification over time. Its most popular variant, the *propositional temporal logic* $\mathcal{PTL}$, is successfully applied in program verification and specification (see e.g. [Manna and Pnueli, 1992; 1995]). $\mathcal{PTL}$-formulas are constructed from propositional variables $p_0, p_1, \ldots$ using the Booleans and the binary *temporal operators* $\mathcal{S}$ ('since') and $\mathcal{U}$ ('until'), the intended meaning of which is as follows:

- $\chi_1 \mathcal{U} \chi_2$ stands for '$\chi_1$ holds true until $\chi_2$ holds;"
- $\chi_1 \mathcal{S} \chi_2$ stands for '$\chi_1$ has been true since $\chi_2$ was true.'

Other temporal connectives like $\Diamond_F$ ('sometime in the future'), $\Box_F$ ('always in the future'), their past counterparts, and $\bigcirc$ ('at the next moment') can be defined via $\mathcal{U}$ and $\mathcal{S}$. For instance, $\Diamond_F \varphi = \top \mathcal{U} \varphi$, $\bigcirc \varphi = \bot \mathcal{U} \varphi$.

To evaluate $\mathcal{PTL}$-formulas in a flow of time $\mathfrak{F} = \langle W, < \rangle$, we have to specify first at which time points the propositional variables hold. Thus, we start with a *valuation* $\mathfrak{V}$ associating with every variable $p$ a subset $\mathfrak{V}(p)$ of $W$. The pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is called a *model* based on the flow of time $\mathfrak{F}$. The *truth-relation* $(\mathfrak{M}, w) \models \varphi$, or simply $w \models \varphi$ if understood (which says that a $\mathcal{PTL}$-formula $\varphi$ holds at moment $w$ in $\mathfrak{M}$) is defined as follows: $w \models p_i$ iff $w \in \mathfrak{V}(p_i)$, $w \models \varphi \wedge \psi$ iff $w \models \varphi$ and $w \models \psi$, $w \models \neg\varphi$ iff $w \not\models \varphi$, and

$$w \models \varphi \mathcal{S} \psi \text{ iff there is } v < w \text{ such that } v \models \psi \text{ and } u \models \varphi \text{ for all } u \in (v, w),$$
$$w \models \varphi \mathcal{U} \psi \text{ iff there is } v > w \text{ such that } v \models \psi \text{ and } u \models \varphi \text{ for all } u \in (w, v).$$

A formula $\varphi$ is *satisfiable* in a class $\mathcal{C}$ of flows of time if there is a model based on a flow of time in $\mathcal{C}$ and a time point $w$ in it such that $w \models \varphi$.

The following results are due to [Sistla and Clarke, 1985; Gabbay *et al.*, 1994; Reynolds, 2001a; Reynolds, 2001b]:

**Theorem 15.** *The satisfiability problem for $\mathcal{PTL}$-formulas is PSPACE-complete in any of the classes mentioned in Theorem 14.*

By comparing the complexity results in Theorems 15 and 14, one might conclude that the propositional temporal language is less expressive than the first-order language $\mathcal{L}^<$. Surprisingly enough, this is not the case: while $\mathcal{L}^<$ is considerably more succinct than $\mathcal{PTL}$, nevertheless the languages turn out to have the same expressive power over many flows of time. Obviously, every $\mathcal{PTL}$-formula $\varphi$ is expressible as an $\mathcal{L}^<$-formula $ST(\varphi)$, called the *standard translation* of $\varphi$ (see [van Benthem, 1983]). The following result is known as the (generalized) *Kamp theorem*; for proofs and more details see [Kamp, 1968; Gabbay *et al.*, 1994].

**Theorem 16.** *The languages $\mathcal{PTL}$ and $\mathcal{L}^<$ have the same expressive power over the flows of time $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Z}, < \rangle$, or $\langle \mathbb{R}, < \rangle$. More precisely, for every $\mathcal{L}^<$-formula $\psi$ with at most one free variable, there is a $\mathcal{PTL}$-formula $\varphi$ such that $\psi$ and $ST(\varphi)$ are equivalent in all models based on any Dedekind complete flow of time.*

## 3.2 Branching time

The formalisms considered so far are not able to express the following statements (due to Aristotle):

- it is necessary that there will be a sea-battle tomorrow;

- it is possible that there will be a sea-battle tomorrow.

Our languages can only say

- $\bigcirc$*sea-battle*, i.e., there will be a sea-battle tomorrow,

they do not distinguish between possible, actual, or necessary future developments. A natural way to formalize assertions of this sort is to add the modal operators $\Box$ and $\Diamond$ to the temporal language and understand them as quantifiers over 'possible histories.' For example, by interpreting $\Diamond$ as 'it is possible that' and $\Box$ as 'it is necessary that,' we can express two Aristotle's statements by the formulas $\Box\bigcirc$*sea-battle* and $\Diamond\bigcirc$*sea-battle*, respectively.

Numerous extensions of $\mathcal{PTL}$ by means of such kind of modal operators have been introduced in different disciplines, say, computer science and AI [Lamport, 1980; Clarke and Emerson, 1981; Emerson and Halpern, 1986] or philosophy [Prior, 1968] (for more references and discussions see [Thomason, 1984; Gabbay *et al.*, 2000]). Here we outline the essential ideas using the simple modal extension of $\mathcal{PTL}$ with $\Box$ and $\Diamond$; it will be called $\mathcal{MPTL}$.

Having fixed the language, we need to choose time structures that could allow for non-trivial interpretations. Clearly, if the flow of time is linear then at every moment the future is fixed, and so $\Box\varphi$ is equivalent to $\varphi$. The flows of time we need should be able to represent different evolutions of history. Since, on the other hand, it is natural to assume that, in contrast to the future, the past is fixed, *trees* as defined below appear to be perfect structures for modelling different histories.

**Definition 17 (branching time model).** A *tree* is a flow of time $\mathfrak{F} = \langle W, < \rangle$ containing a point $r$, called the *root* of $\mathfrak{F}$, for which $W = \{v : r < v\} \cup \{r\}$, and such that for every $w \in W$, the set $\{w : v < w\}$ is finite and linearly ordered by $<$.[7] A *history* in $\mathfrak{F}$ is a maximal linearly $<$-ordered subset of $W$.

A *branching time model* is a structure $\mathfrak{B} = \langle \mathfrak{F}, \mathcal{H}, \mathfrak{V} \rangle$, where $\mathfrak{F} = \langle W, < \rangle$ is a tree, $\mathcal{H}$ a set of histories in $\mathfrak{F}$—the set of possible flows of time in the model—and $\mathfrak{V}$ is a *valuation* in $\mathfrak{F}$. Formulas are evaluated relative to pairs $(h, w)$ consisting of an *actual history* $h \in \mathcal{H}$ and a time point $w \in h$. In such a pair $(h, w)$, the temporal operators are interpreted along the actual history $h$ as in the linear time framework, while the modal operators quantify over the set of all histories $\mathcal{H}(w) = \{h' \in \mathcal{H} : w \in h'\}$ coming through $w$. More precisely, the *truth-relation* $\models$ between pairs $(h, w)$ and $\mathcal{MPTL}$-formulas $\varphi$ is defined inductively in the following way (we omit the clauses for the Booleans):

- $(h, w) \models p$ iff $w \in \mathfrak{V}(p)$;
- $(h, w) \models \varphi \mathcal{U} \psi$ iff there is $v \in h$ such that $v > w$, $(h, v) \models \psi$ and $(h, u) \models \varphi$ for all $u \in (w, v)$;
- $(h, w) \models \varphi \mathcal{S} \psi$ iff there is $v \in h$ such that $v < w$, $(h, v) \models \psi$ and $(h, u) \models \varphi$ for all $u \in (v, w)$;
- $(h, w) \models \Diamond\varphi$ iff there is $h' \in \mathcal{H}(w)$ such that $(h', w) \models \varphi$;
- $(h, w) \models \Box\varphi$ iff $(h', w) \models \varphi$ for all $h' \in \mathcal{H}(w)$.

[7] Other definitions of trees can be more liberal not requiring the finitness of $\{w : v < w\}$. One can develop the whole formalism in this more general framework; see [Thomason, 1984].

Note that propositional variables are assumed to have no temporal aspect—their truth-values at $(h, w)$ do not depend on the actual history $h$.

The branching time model defined above reflects the Ockhamist view of time. We refer the reader to [Burgess, 1979; Zanardo, 1996; Gabbay *et al.*, 2000; Reynolds, 2002] for more information about this and related approaches. Note only a close connection to the computational tree logics CTL and CTL* that are widely used in program verification and specification [Clarke and Emerson, 1981; Emerson and Halpern, 1986; Clarke *et al.*, 2000].

It might seem more natural to quantify with $\Diamond$ and $\Box$ over the set of *all* histories in the tree rather than its subset $\mathcal{H}$. But then we would be forced to accept possibly unintended histories in $\mathfrak{F}$ as possible flows of time. Here is an example of a formula satisfiable in a branching time model as defined above, but not in a branching time model in which $\mathcal{H}$ is the set of all histories. The formula is a conjunction of the following three $\mathcal{MPTL}$-formulas:

$$\Box \mathsf{P}(Kosovo, Yugoslavia),$$
$$\Box \Diamond_F \Box_F \mathsf{EC}(Kosovo, Yugoslavia),$$
$$\Box \Box_F \big(\mathsf{P}(Kosovo, Yugoslavia) \to \Diamond \bigcirc \mathsf{P}(Kosovo, Yugoslavia)\big).$$

The first formula means that in all histories, at present Kosovo is part of Yugoslavia. The second says that in all possible histories, there'll be a time starting from which Kosovo will be externally connected to Yugoslavia. And the last formula claims that in all possible histories, it is always the case that if Kosovo is part of Yugoslavia then it is still possible that it will remain in Yugoslavia at least one more day. (Since we do not have a combined spatio-temporal language yet, the RCC-8 predicates $\mathsf{P}(Kosovo, Yugoslavia)$ and $\mathsf{EC}(Kosovo, Yugoslavia)$ should be regarded as a propositional variable and its negation, respectively.)

The following results are due to [Burgess, 1979]. Is it so? Or Rabin. Complexity??

**Theorem 18.** *The satisfiability problem for $\mathcal{PTLM}$-formulas is decidable.*

## 3.3 First-order temporal logic

So far, we haven't endowed time points with any structures that could represent states of application domains (e.g. spatial knowledge bases) at these points. When doing these, we get into the realm of *first-order temporal logic* or its variants, say, *temporal description logic* (see e.g. [Wolter and Zakharyaschev, 2000c]).

Suppose that in order to represent our application domain we use a first-order language $\mathcal{FO}$ with predicates $P_0, P_1, \dots$ of some fixed arity. Assume also that the intended flow of time $\mathfrak{F} = \langle W, < \rangle$ is linear. Then a *first-order temporal model based on* $\mathfrak{F}$ is a pair of the form $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{m} \rangle$, where, for each $w \in W$,

$$\mathfrak{m}(w) = \langle D, P_0^w, P_1^w \dots \rangle \tag{5}$$

is an ordinary $\mathcal{FO}$-structure, i.e., $D$ is a non-empty set and the $P_i^w$ are relations on $D$ of the same arity as $P_i$. Note that the $P_i^w$ depend on $w$, while the domain $D$ of $\mathfrak{m}(w)$ is assumed to be constant. Models of this type are often called *models with constant domains.*

An appropriate language for speaking about such models is the combination of $\mathcal{FO}$ with $\mathcal{PTL}$ in which the temporal operators $\mathcal{S}$ and $\mathcal{U}$ can be applied to first-order formulas. It will be denoted by $\mathcal{FOTL}$. The temporal operators $\mathcal{S}$ and $\mathcal{U}$ take care of the temporal dimension, while the first-order part of the language allows us to speak about the domain dimension.

To define the *truth-relation* $\models$ between time points and formulas, we first fix an *assignment* $\mathfrak{a}$ associating elements in $D$ to individual variables. Then $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ is defined by taking

- $(\mathfrak{M}, w) \models^{\mathfrak{a}} P_i(x_1, \ldots, x_k)$ iff $\mathfrak{m}(w) \models P_i[\mathfrak{a}(x_1), \ldots, \mathfrak{a}(x_k)]$,
- $(\mathfrak{M}, w) \models^{\mathfrak{a}} \exists x \varphi$ iff there exists an assignment $\mathfrak{b}$ which may differ from $\mathfrak{a}$ only on $x$ and such that $(\mathfrak{M}, w) \models^{\mathfrak{b}} \varphi$,

and the propositional clauses for the Booleans and temporal operators. Unfortunately, the resulting logics turn out to be highly undecidable for most important flows of time. In particular, we have the following result due to Scott and Lindström (unpublished); for a proof see e.g. [Gabbay *et al.*, 1994]):

**Theorem 19.** *The satisfiability problem for $\mathcal{FOTL}$-formulas in models based on $\langle \mathbb{R}, < \rangle$, $\langle \mathbb{Z}, < \rangle$, or $\langle \mathbb{N}, < \rangle$ is not recursively enumerable.*

Moreover, even seemingly simple fragments, such as the two-variable fragment of $\mathcal{FOTL}$ (containing formulas with the variables $x, y$ only) and the monadic fragment of $\mathcal{FOTL}$ (containing fomulas with unary predicates only), are undecidable in any natural class of flows of time [Merz, 1992; Hodkinson *et al.*, 2000]. These 'negative' results have been a serious obstacle for applying first-order temporal logic in computer science and AI.

A certain breakthrough has been recently achieved in [Hodkinson *et al.*, 2000; Wolter and Zakharyaschev, 2001], where a so-called *monodic fragment* of $\mathcal{FOTL}$ was shown to have a much better computation behavior. The monodic fragment consists of those $\mathcal{FOTL}$-formulas that do not contain a subformula starting with $\mathcal{S}$ or $\mathcal{U}$ and having more than one free variable. Unlike the full $\mathcal{FOTL}$, the set of monodic formulas valid in models based on $\langle \mathbb{N}, < \rangle$ turns out to be axiomatizable. Various decidable subfragments of the monodic fragment are described in [Hodkinson *et al.*, 2000; Wolter and Zakharyaschev, 2001]. In particular, the following results will be used later on in this paper:

**Theorem 20.** (i) *Let $\mathcal{C}$ be one of the following classes of flows of time: the class of all strict linear orders, $\{\langle \mathbb{Q}, < \rangle\}$, $\{\langle \mathbb{Z}, < \rangle\}$, $\{\langle \mathbb{N}, < \rangle\}$. Then the satisfiability problem for the one-variable fragment of $\mathcal{FOTL}$ in models based on flows of time in $\mathcal{C}$ is decidable.*

(ii) *Let $\mathcal{C}^+$ be one of the classes mentioned above or $\{\langle \mathbb{R}, < \rangle\}$. Then the satisfiability problem for the one-variable fragment of $\mathcal{FOTL}$ in models based on flows of time in $\mathcal{C}^+$ and having finite first-order domains is decidable.*

(iii) *In both cases, the satisfiability problem in models based on $\langle \mathbb{N}, < \rangle$ or $\langle \mathbb{Z}, < \rangle$ is EXPSPACE-complete.*

(The complexity of satisfiability in the flows of time different from $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$ remains an open problem.)

Let us now turn to branching time. Given a tree $\mathfrak{F} = \langle W, < \rangle$, a set of histories $\mathcal{H}$ in $\mathfrak{F}$, and a function $\mathfrak{m}$ of the form (5), we can form the *first-order* *branching time model* $\mathfrak{M} = \langle \mathfrak{F}, \mathcal{H}, \mathfrak{m} \rangle$. Having fixed an assignment $\mathfrak{a}$ in $D$, we define the truth-relation $\models$ between pairs $(h, w)$ and formulas $\varphi$ of the *first-order branching temporal logic* $\mathcal{FOBTL}$ by taking

- $(h, w) \models^{\mathfrak{a}} \Box\varphi$ iff $(h', w) \models^{\mathfrak{a}} \varphi$ for all $h' \in \mathcal{H}(w)$

and keeping the other inductive clauses similar to the linear case. The resulting logic is at least as complex as $\mathcal{FOTL}$ on $\langle \mathbb{N}, < \rangle$; hence it is highly undecidable. However, again the monodic fragment (consisting of all $\mathcal{FOBTL}$-formulas in which none of the temporal and modal operators has more than one free variable in its scope) provides us with ways of obtaining decidable fragments [?]. In particular, the following holds: Ref?

**Theorem 21.** (i) *The satisfiability problem for the one-variable fragment of* $\mathcal{FOBTL}$ *is decidable.*

(ii) *The satisfiability problem for the one-variable fragment of* $\mathcal{FOBTL}$ *in models with finite first-order domains and finite sets of histories is decidable.* Complexity?

## 3.4 Interval temporal logic

Similar to RCC-8, instead of time points one can take extended time entities, i.e., intervals, as primitives. This approach to temporal representation and reasoning reflects the fact that certain assertions can be evaluated only at periods of time (e.g. 'John often drinks beer'). It was developed by Allen [1983; 1984], who observed, in particular, that relative positions of any two intervals $i$ and $j$ of a strict linear order can be described by precisely one of the thirteen basic interval relations: $\mathsf{before}(i,j)$, $\mathsf{meets}(i,j)$, $\mathsf{overlaps}(i,j)$, $\mathsf{during}(i,j)$, $\mathsf{starts}(i,j)$, $\mathsf{finishes}(i,j)$, their inverses (i.e., $\mathsf{before}(j,i)$, $\mathsf{meets}(j,i)$, etc.), and $\mathsf{equal}(i,j)$. Let us denote by $\mathcal{A}\ell\ell$-13 the language whose alphabet contains the thirteen binary predicate symbols as above, *interval variables* $i$, $j$, etc., and the Booleans. Formulas of $\mathcal{A}\ell\ell$-13 are just Boolean combinations of the basic predicates.

To provide a semantics for $\mathcal{A}\ell\ell$-13 formulas, suppose that the flow of time is a strict linear order $\mathfrak{F} = \langle W, < \rangle$. An *assignment* in $\mathfrak{F}$ is a function $\mathfrak{a}$ mapping the interval variables into *temporal intervals in* $\mathfrak{F}$. There may be different views on what the temporal intervals in $\mathfrak{F}$ should be. We take perhaps the most 'liberal' version by defining them as arbitrary non-empty convex sets in $\mathfrak{F}$. In other words, a temporal interval $\mathfrak{a}(i)$ in $\mathfrak{F}$ is a non-empty subset of $W$ such that

$$\forall x, y \in \mathfrak{a}(i)\ \forall z \in W (x < z < y \to z \in \mathfrak{a}(i)).$$

The truth-relation $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ for atomic $\mathcal{A}\ell\ell$-13 formulas is defined in the natural way. For instance,

$$\begin{array}{lll}
\mathfrak{F} \models^{\mathfrak{a}} \mathsf{meets}(i,j) & \text{iff} & \forall x, y\ (x \in \mathfrak{a}(i) \wedge y \in \mathfrak{a}(j) \to x < y \wedge \forall z (x < z < y \\
& & \quad \to\ z \in \mathfrak{a}(i) \vee z \in \mathfrak{a}(j))), \\
\mathfrak{F} \models^{\mathfrak{a}} \mathsf{overlaps}(i,j) & \text{iff} & \mathfrak{a}(i) \cap \mathfrak{a}(j) \neq \emptyset \wedge \exists x, y\ (x < y \\
& & \quad \wedge\ x \in \mathfrak{a}(j) \wedge x \notin \mathfrak{a}(i) \wedge y \in \mathfrak{a}(j) \wedge y \notin \mathfrak{a}(i)), \\
\mathfrak{F} \models^{\mathfrak{a}} \mathsf{starts}(i,j) & \text{iff} & \mathfrak{a}(i) \subseteq \mathfrak{a}(y) \wedge \mathfrak{a}(i) \neq \mathfrak{a}(j) \wedge \forall x, y\ (x < y \\
& & \quad \wedge\ x \in \mathfrak{a}(j) \wedge y \in \mathfrak{a}(i) \cap \mathfrak{a}(j) \to x \in \mathfrak{a}(i)), \\
\mathfrak{F} \models^{\mathfrak{a}} \mathsf{during}(i,j) & \text{iff} & \exists x, y, z\ (x < y < z \wedge x \in \mathfrak{a}(j) \wedge x \notin \mathfrak{a}(i) \\
& & \quad \wedge\ y \in \mathfrak{a}(i) \wedge z \in \mathfrak{a}(j) \wedge z \notin \mathfrak{a}(i)).
\end{array}$$

We say that $\varphi$ *is satisfiable in a class* $\mathcal{C}$ of flows of time if $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ holds for some $\mathfrak{F} \in \mathcal{C}$ and assignment $\mathfrak{a}$ in $\mathfrak{F}$.

Usually $\mathcal{A}\ell\ell$-13 serves as a basis for more complex languages which, besides temporal constraints, use other predicates such as $\mathsf{HOLDS}(\phi, i)$ (property $\phi$ holds during interval $i$), $\mathsf{OCCUR}(e, i)$ (event $e$ happens over interval $i$). Some examples will be provided in Section 4.3.

The following result was shown in [van Beek *et al.*, 1986]:

**Theorem 22.** *The satisfiability problem for $\mathcal{A}\ell\ell$-13 formulas in any class of linear flows of time is NP-complete.*

Note also that $\mathcal{A}\ell\ell$-13 can be easily embedded into point-based temporal logic; for details see [Blackburn, 1993].

# 4 Of space and time

Following our semantical approach, we start designing logics of time *and* space by defining their intended models—spatio-temporal structures—as a combination of topological and temporal models. We consider first the linear point-based paradigm.

## 4.1 Spatio-temporal logics: linear point-based time

**Definition 23 (topological temporal model).** A *topological temporal model* (or *tt-model*, for short) based on a topological model $\mathfrak{S}$ of the form (4) and a flow of time $\mathfrak{F} = \langle W, < \rangle$ is simply the pair $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{F} \rangle$. An *assignment* in $\mathfrak{M}$ is a function $\mathfrak{a}$ associating with each region variable $X$ and each moment of time $w \in W$ a set $\mathfrak{a}(X, w) \in \mathcal{R}(\mathfrak{T})$, the *state* of $X$ at $w$.

Thus, tt-models can be regarded as *two-dimensional* structures. Having fixed a moment of time, we can move in the 'spatial dimension' representing the states of regions at this moment. Having fixed a spatial region, we can move along the 'temporal dimension' tracing the evolution of this region in time. (Note the difference from first-order temporal models in which the values of individual variables are constant over time, while the extensions of predicate symbols can vary.)

Let us turn now to the syntactical parameters of spatio-temporal hybrids.

### 4.1.1 Quantification over regions

Unfortunately, quantification over region variables in tt-models—even for extremely weak languages—results in undecidable or non-axiomatizable logics. We show here only one example. Consider the first-order spatio-temporal language $\mathcal{FOST}$ based on the following alphabet:

- an infinite set of *local region variables* $X_0, X_1, \ldots$;
- an infinite set of *global region variables* $Y_0, Y_1, \ldots$;
- the binary temporal operator $\mathcal{U}$ ('until');
- the binary predicate $\mathsf{EQ}(Z_1, Z_2)$;

$\mathcal{FOST}$*-formulas* are defined as follows:

- $\mathsf{EQ}(Z_1, Z_2)$ is an atomic formula, where $Z_1$, $Z_2$ are region variables;
- if $\varphi$ and $\psi$ are formulas and $X$ is a *global* region variable, then $\neg\varphi$, $\varphi \wedge \psi$, and $\forall X \varphi$ are formulas.

The difference between local and global region variables is that the former range over 'mobile' regions, while the latter denote regions that are supposed to be immovable. Thus, an assignment $\mathfrak{a}$ in a tt-model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{F} \rangle$ should be such that $\mathfrak{a}(Y_i, u) = \mathfrak{a}(Y_i, v)$ for any time points $u$ and $v$ in $\mathfrak{F}$ and any global variable $Y_i$. The definition of the truth-relation must be clear: we put

$$(\mathfrak{M}, w) \models^{\mathfrak{a}} \mathsf{EQ}(Z_1, Z_2) \quad \text{iff} \quad \mathfrak{a}(Z_1, w) = \mathfrak{a}(Z_2, w)$$

and define the Booleans, quantifiers, and the temporal operator in the standard way.

Thus, in this language we can reason about the equality of regions over time, but nothing else. The language looks completely 'harmless.' And yet, the following is easily derived from results of [Merz, 1992]:

**Theorem 24.** *The satisfiability problem for $\mathcal{FOST}$-formulas in tt-models based on infinite flows of time is undecidable; it is not even recursively enumerable for the flows $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$.*

The reason explaining such a 'bad' computational behavior is the *interaction* between the temporal operator and the quantifiers over region variables, which is similar to the interaction between the compass operators in Example 1. Again we are forced to omit quantification and take BRCC-8 as the spatial component of the spatio-temporal logics to be constructed.

### 4.1.2 Spatio-temporal representation based on BRCC-8

In this section, we construct three spatio-temporal logics based on BRCC-8. We denote them by $\mathcal{ST}_0$–$\mathcal{ST}_2$.

$\mathcal{ST}_0$**.** The simplest one allows applications of the temporal operators $\mathcal{S}$ and $\mathcal{U}$ only to BRCC-8 formulas. More precisely, the *spatio-temporal language* $\mathcal{ST}_0$ is defined as follows. Every formula of BRCC-8 is also an $\mathcal{ST}_0$-formula, and if $\varphi$ and $\psi$ are $\mathcal{ST}_0$-formulas then so are $\varphi\mathcal{S}\psi$, $\varphi\mathcal{U}\psi$, $\varphi \wedge \psi$, and $\neg\varphi$. As usual, we use the abbreviations $\bigcirc\varphi = \bot\mathcal{U}\varphi$, $\Diamond_F\varphi = \top\mathcal{U}\varphi$, $\Box_F\varphi = \neg\Diamond_F\neg\varphi$; a new one is $\varphi\mathcal{W}\psi = \Box_F\varphi \vee (\varphi\mathcal{U}\psi)$, where $\mathcal{W}$ stands for 'waiting for' (it is also known as 'unless;' see [Manna and Pnueli, 1992]).

For a tt-model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{F} \rangle$, an assignment $\mathfrak{a}$ in it, an $\mathcal{ST}_0$-formula $\varphi$, and a time point $w$ in $\mathfrak{F}$, define the *truth-relation* $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ by induction on the construction of $\varphi$. Let $\mathfrak{a}_w$ be the assignment in $\mathfrak{S}$ defined by $\mathfrak{a}_w(X) = \mathfrak{a}(X, w)$, for every region variable $X$. Now,

- if $\varphi$ contains no temporal operator, then $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ iff $\mathfrak{S} \models^{\mathfrak{a}_w} \varphi$;
- $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi\mathcal{U}\psi$ iff there is $v > w$ such that $(\mathfrak{M}, v) \models^{\mathfrak{a}} \psi$ and $(\mathfrak{M}, u) \models^{\mathfrak{a}} \varphi$ for every $u$ in the interval $w < u < v$;

- $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi \mathcal{S} \psi$ iff there is $v < w$ such that $(\mathfrak{M}, v) \models^{\mathfrak{a}} \psi$ and $(\mathfrak{M}, u) \models^{\mathfrak{a}} \varphi$ for every $u$ in the interval $v < u < w$.

The interaction between time and space in $\mathcal{ST}_0$ is rather weak. In fact, satisfiability of $\mathcal{ST}_0$-formulas in a given infinite flow of time $\mathfrak{F}$ is easily, but exponentially, reducible to satisfiability of $\mathcal{PTL}$-formulas in $\mathfrak{F}$. (As we saw in Section 2.2.3, BRCC-8 is reducible to S5, which, in turn, can be exponentially reduced to $\mathcal{PTL}$.) Moreover, for $\langle \mathbb{N}, < \rangle$ a PSPACE satisfiability checking algorithm was constructed in [Wolter and Zakharyaschev, 2000b]. To sum up, using Theorem 15, we obtain:

**Theorem 25.** *Let $\mathcal{C}^+$ be one of the classes defined in Section 3.3. Then the satisfiability problem for $\mathcal{ST}_0$-formulas in tt-models based on flows of time in $\mathcal{C}^+$ is decidable in EXPSPACE. For $\langle \mathbb{N}, < \rangle$ it is PSPACE-complete.*

It is an open problem whether satisfiability of $\mathcal{ST}_0$-formulas in flows different from $\langle \mathbb{N}, < \rangle$ can be checked in PSPACE as well.

The language $\mathcal{ST}_0$ is expressive enough to capture some aspects of *continuity of changes* (see e.g. [Cohn, 1997]):

$$
\begin{aligned}
\mathsf{DC}(X,Y) &\to \mathsf{DC}(X,Y)\,\mathcal{W}\,\mathsf{EC}(X,Y),\\
\mathsf{EC}(X,Y) &\to \mathsf{EC}(X,Y)\,\mathcal{W}\,(\mathsf{DC}(X,Y) \vee \mathsf{PO}(X,Y)),\\
\mathsf{PO}(X,Y) &\to \mathsf{PO}(X,Y)\,\mathcal{W}\,(\mathsf{EC}(X,Y)\ \vee\\
&\qquad\qquad \mathsf{TPP}(X,Y) \vee \mathsf{EQ}(X,Y) \vee \mathsf{TPPi}(X,Y)),\\
&\text{etc.}
\end{aligned}
$$

The first of these formulas, for instance, says that if two regions are disconnected at some moment, then either they will remain disconnected forever or they are disconnected until they become externally connected. If the flow of time is discrete then these conditions are equivalent to:

$$
\begin{aligned}
\mathsf{DC}(X,Y) &\to \bigcirc(\mathsf{DC}(X,Y) \vee \mathsf{EC}(X,Y)),\\
\mathsf{EC}(X,Y) &\to \bigcirc(\mathsf{EC}(X,Y) \vee \mathsf{DC}(X,Y) \vee \mathsf{PO}(X,Y)),\\
\mathsf{PO}(X,Y) &\to \bigcirc(\mathsf{PO}(X,Y) \vee \mathsf{EC}(X,Y)\ \vee\\
&\qquad\qquad \mathsf{TPP}(X,Y) \vee \mathsf{EQ}(X,Y) \vee \mathsf{TPPi}(X,Y)),\\
&\text{etc.}
\end{aligned}
$$

However, the expressive power of $\mathcal{ST}_0$ is rather limited. In particular, we can compare regions only at one moment of time, but we are not able to connect a region as it is 'today' with its state 'tomorrow' to say, for example, that it is expanding or remains the same. In other words, we can express the dynamics of relations between regions, say,

$$\neg \Box_F \mathsf{P}(Kosovo, Yugoslavia)$$

('it is not true that Kosovo will always be part of Yugoslavia'), but not the dynamics of regions themselves, for instance, that

$$\Box_F \mathsf{P}(EU, \bigcirc EU),$$

where $\bigcirc EU$ at moment $n$ intends to denote the space occupied by the EU at the next moment (so for the flow of time $\langle \mathbb{N}, < \rangle$, the last formula means: 'the

EU will never contract'). This new constructor may also be important to refine the continuity assumption by requiring that

$$\Box_F(\mathsf{EQ}(X, \bigcirc X) \vee \mathsf{O}(X, \bigcirc X)),$$

i.e., 'regions $X$ and $\bigcirc X$ either coincide or overlap.'

$\mathcal{ST}_1$. To capture this dynamics, we extend $\mathcal{ST}_0$ by allowing applications of the next-time operator $\bigcirc$ not only to formulas but also to Boolean region terms. Thus, arguments of RCC-8 predicates can be now arbitrary $\bigcirc$-*terms* which are constructed from region variables using the Booleans and $\bigcirc$. For instance, $\bigcirc \bigcirc X$ represents region $X$ as it will be 'the day after tomorrow.' Denote the resulting language by $\mathcal{ST}_1$, and let $\mathcal{ST}_1'$ be its sublanguage with only one temporal operator $\bigcirc$ ($\mathcal{S}$ and $\mathcal{U}$ are not allowed). Obviously, $\mathcal{ST}_1$ is more expressive than $\mathcal{ST}_0$ only for *discrete* flows of time; in dense flows like $\langle \mathbb{Q}, < \rangle$ or $\langle \mathbb{R}, < \rangle$ the 'next-time' operator makes no sense. If $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{F} \rangle$ is a tt-model, $\mathfrak{a}$ an assignment in it, and $t$ a $\bigcirc$-term, then we put

$$\mathfrak{a}(\bigcirc t, w) = \begin{cases} \mathfrak{a}(t, w') & \text{if } w' \text{ is an immediate successor of } w \text{ in } \mathfrak{F}, \\ \emptyset & \text{if } w \text{ has no immediate successor in } \mathfrak{F}. \end{cases}$$

**Theorem 26.** (i) *The satisfiability problem for $\mathcal{ST}_1$-formulas in tt-models based on flows of time in $\mathcal{C}$ is decidable; for $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$ it is decidable in EXPSPACE.*

(ii) *The satisfiability problem for $\mathcal{ST}_1'$-formulas in tt-models based on $\langle \mathbb{N}, < \rangle$ is NP-complete.*

The EXPSPACE-upper bound and (ii) are proved in [Wolter and Zakharyaschev, 2000b]. A proof of (i) based on an embedding into first-order temporal logic is sketched below and given in detail in [Gabbay *et al.*, 2001]. (The lower bound is still unknown.)

Using $\mathcal{ST}_1$ we can express in $\langle \mathbb{N}, < \rangle$ that region $X$ will always be the same, i.e., $X$ is global (or rigid):

$$\Box_F \mathsf{EQ}(X, \bigcirc X),$$

or that it has at most two distinct states, one on 'even days,' another on 'odd ones:'

$$\Box_F \mathsf{EQ}(X, \bigcirc \bigcirc X).$$

Note, by the way, that the $\mathcal{ST}_1$-formula

$$\Box_F \mathsf{NTPP}(X, \bigcirc X)$$

is satisfiable only in models based on infinite topological spaces—unlike BRCC-8 formulas, for which finite topological spaces are enough (see Theorem 10).

It may appear that $\mathcal{ST}_1$ is able to compare regions only within fixed time intervals. However, using an auxiliary global variable $X$ we can write, for instance,

$$\Box_F \mathsf{EQ}(X, \bigcirc X) \wedge \Diamond_F \mathsf{EQ}(X, EU) \wedge \mathsf{P}(Russia, X).$$

This formula is satisfiable iff 'some day in the future the *present* territory of Russia will be part of the EU.' Note that the formula

$$\Diamond_F \mathsf{P}(Russia, EU)$$

means that there will be a day when Russia—its territory on that day (say, without Chechnya but with Byelorussia)—becomes part of the EU.

Imagine now that we want to express in our spatio-temporal language that all countries in Europe will pass through the euro-zone, but only Germany (in its present territory) will use the euro forever. Unfortunately, we don't know which countries will be formed in Europe in the future, so we can't simply write down all formulas of the form

$$\Diamond_F \mathsf{P}(X, \textit{Euro-zone}).$$

What we actually need is the possibility of constructing regions $\Diamond_F X$ and $\Box_F X$ which contain all the points that will belong to region $X$ in the future and only common points of all future states of $X$, respectively. Then we can write:

$$\mathsf{EQ}(\textit{Europe}, \Diamond_F \textit{Euro-zone}) \quad \text{and} \quad \mathsf{EQ}(\textit{Germany}, \Box_F \textit{Euro-zone}).$$

The formula $\mathsf{P}(\textit{Russia}, \Diamond_F \textit{EU})$ says that all points of the present territory of Russia will belong to the EU in the future (but perhaps at different moments of time).

$\mathcal{ST}_\mathbf{2}$. So let us extend $\mathcal{ST}_0$ by allowing the use of *temporal region terms*, constructed from region variables, the Booleans, and the temporal operators $\mathcal{U}$ and $\mathcal{S}$ with all their derivatives, as arguments of the RCC-8 predicate. The resulting language will be denoted by $\mathcal{ST}_2$. The intended semantics of temporal region terms is as follows. Suppose $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{F} \rangle$ is a tt-model and $\mathfrak{a}$ an assignment in it. Define inductively the *value* $\mathfrak{a}(t, w)$ of a temporal region term $t$ under $\mathfrak{a}$ at $w$ in $\mathfrak{M}$ by taking:

$$\begin{aligned}
\mathfrak{a}(\Diamond_F t, w) &= \mathbb{CI} \bigcup_{v>w} \mathfrak{a}(t, v), \\
\mathfrak{a}(\Box_F t, w) &= \mathbb{CI} \bigcap_{v>w} \mathfrak{a}(t, v), \\
\mathfrak{a}(t_1 \mathcal{U} t_2, w) &= \mathbb{CI}\, \{x : \exists v > w\, (x \in \mathfrak{a}(t_2, v) \wedge \forall u\, (w < u < v \to x \in \mathfrak{a}(t_1, u)))\}, \\
\mathfrak{a}(t_1 \mathcal{S} t_2, w) &= \mathbb{CI}\, \{x : \exists v < w\, (x \in \mathfrak{a}(t_2, v) \wedge \forall u\, (w > u > v \to x \in \mathfrak{a}(t_2, u)))\},
\end{aligned}$$

and the corresponding clauses for $\Diamond_P$ and $\Box_P$. For example, the formula

$$\mathsf{DC}(\textit{Russia}\ \mathcal{S}\ \textit{Russian\_Empire}, \textit{Russia}\ \mathcal{S}\ \textit{Germany})$$

can be used to say that the part of Russia that has been remaining Russian since 1917 is not connected to the part of Germany (Königsberg) that became Russian after the Second World War.

We remind the reader that we have to use the prefix $\mathbb{CI}$ in the right-hand parts of the definition above because infinite unions and intersections of regular closed sets are not necessarily regular closed (see (1); however, this is the case for models based on Kripke spaces), while all temporal region terms are supposed to be interpreted by 'regions' of topological spaces.[8] Actually, as we shall see

[8]It is also worth noting that the operators $\Diamond_F$ and $\Box_F$ on temporal region terms are dual in the sense that for every assignment $\mathfrak{a}$, every term $t$, and every moment $w$ we have $\mathfrak{a}(\Diamond_F t, w) = \mathfrak{a}(\neg\Box_F\neg t, w)$.

below, infinite operations bring various semantical complications. To avoid this problem we can try to restrict assignments in models in such a way that infinite intersections and unions can be reduced to finite ones. There are different ways of doing this. One idea would be to accept the *Finite Change Assumption*:

**FCA** *No region can change its spatial configuration infinitely often.*

This means that under **FCA** we consider only those assignments $\mathfrak{a}$ in tt-models $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{F} \rangle$ that satisfy the following condition: for every temporal region term $t$ there are pairwise disjoint convex sets $I_1, \dots, I_n$ of points in $\mathfrak{F} = \langle W, < \rangle$ such that $W = I_1 \cup \cdots \cup I_n$ and the state of $t$ remains constant on each $I_j$, i.e., $\mathfrak{a}(t, u) = \mathfrak{a}(t, v)$ for every $u, v \in I_j$. Note that for the flow $\mathfrak{F} = \langle \mathbb{N}, < \rangle$, **FCA** can be captured by the $\mathcal{ST}_1$-formulas $\Diamond_F \Box_F \mathsf{EQ}(t, \bigcirc t)$.

Of course, **FCA** excludes some mathematically interesting cases. Yet, it is absolutely adequate for many applications, for example, when we are planning a job which eventually must be completed (consider a robot painting a wall). Optimists would accept **FCA** to describe the geography of Europe in the examples above. In temporal databases the time line is often assumed to be finite, though arbitrarily long, which corresponds to **FCA**. Another, more general, way of reducing infinite unions and intersections to finite ones is to adopt the *Finite State Assumption*:

**FSA** *Every region can have only finitely many possible states (although it may change its states infinitely often).*

Say that a tt-model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{F} \rangle$ with an assignment $\mathfrak{a}$ satisfies **FSA**, or is an **FSA**-*model*, if for every temporal region term $t$ there are finitely many sets $A_1, \dots, A_m \in \mathcal{R}(\mathfrak{T})$ such that $\{\mathfrak{a}(t, w) : w \in W\} = \{A_1, \dots, A_m\}$. These models can be used, for instance, to capture periodic fluctuations due to season or climate changes, say, a daily tide.

**Theorem 27.** (i) *The satisfiability problem for $\mathcal{ST}_2$-formulas in* **FSA**-*models based on flows of time in $\mathcal{C}^+$ is decidable; for $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$ it is decidable in EXPSPACE.*

(ii) *An $\mathcal{ST}_2$-formula is satisfiable in an* **FSA**-*model iff it is satisfiable in an* **FSA**-*model based on a finite topological space.*

For the flow $\langle \mathbb{N}, < \rangle$ this result is proved in [Wolter and Zakharyaschev, 2000b]. A proof of the general result is provided in [Gabbay *et al.*, 2001]. It is based on an embedding into first-order temporal logic and sketched below. The complexity of the satisfiability problem in (i) is unknown.

It is worth noting that, instead of the propositional temporal language $\mathcal{PTL}$, we could have combined with $\mathsf{BRCC}$-8 the first-order language $\mathcal{L}^<$. We would then obtain a *two-sorted* language with variables $t$ of sort 'time' ranging over time points and variables $X$ of sort 'region' ranging over regions in topological spaces. The new ingredient would be eight *ternary* predicates $\mathsf{DC}(t, X, Y)$, $\mathsf{EC}(t, X, Y)$, etc., the intuitive meaning of which is 'at moment $t$, region $X$ is disconnected from region $Y$', etc. Similarly to Kamp's theorem on the expressive equivalence of $\mathcal{L}^<$ and $\mathcal{PTL}$, one can show that certain two-sorted logics have the same expressive power as certain logics in the $\mathcal{ST}_i$-hierarchy.

### 4.1.3 Example

We illustrate possible applications of the language introduced in the previous section by showing a toy spatio-temporal knowledge base. Consider the following scenario of how the foot and mouth epidemic spreads across a country. Assume that the country consists of disjoint regions: farms, towns, forests, rivers, etc. The map of the country can clearly be represented as a database of $\mathsf{RCC\text{-}8}$ formulas. Besides, we require that all these regions are rigid, i.e., $\Box_F^+ \mathsf{EQ}(X, \bigcirc X)$ (as quantification over regions is not allowed, we have to write such formulas for all regions $X$ on the map). Now, suppose that at moment 0 foot and mouth has been detected only at one farm $X_0$:

$$\mathsf{EQ}(\mathit{F\&M}, X_0) \wedge \mathsf{P}(X_0, \mathit{Farm}).$$

The region $\mathit{F\&M}$, representing the current contaminated part of the country, is not rigid. Nor is the region $\mathit{Stock}$ representing the farms with live-stock. Let $X_0, \ldots, X_n$ be all the farms in the country. We then should clearly have, for all $i \leq n$:

$$\begin{array}{l}
\Box_F^+ \big(\mathsf{O}(X_i, \mathit{Stock}) \to \mathsf{P}(X_i, \mathit{Stock})\big). \\
\Box_F^+ \mathsf{P}(\mathit{Stock}, X_0 \sqcup \cdots \sqcup X_n). \\
\Box_F^+ \big((\mathsf{O}(X_i, F\&M) \to \mathsf{P}(X_i, \mathit{F\&M})\big). \\
\Box_F^+ \mathsf{P}(\mathit{F\&M}, \mathit{Stock}).
\end{array}$$

Suppose also that if one farm suffers from foot and mouth, then at the next moment the disease will spread to all neighboring farms with stock, but not further, i.e., for all $i, j \leq n$,

$$\begin{array}{l}
\Box_F^+ \big(\mathsf{P}(X_i, \mathit{F\&M}) \wedge \mathsf{EC}(X_i, X_j) \wedge \mathsf{P}(X_j, \mathit{Stock}) \to \bigcirc \mathsf{P}(X_j, \mathit{F\&M})\big). \\
\Box_F^+ \big(\neg \mathsf{EC}(X_i, \mathit{F\&M}) \to \bigcirc \neg \mathsf{P}(X_i, \mathit{F\&M})\big).
\end{array}$$

As the government takes proper measures against the disease, in a few moments (say, two for definiteness), a farm with foot and mouth will have no live-stock. On the other hand, the government is going to help the farmers to continue their business, so eventually new stock will be purchased (but nobody knows when):

$$\begin{array}{l}
\Box_F^+ \Big(\mathsf{P}(X_i, \mathit{F\&M}) \to \bigcirc \bigcirc \big(\neg \mathsf{O}(X_i, \mathit{F\&M}) \wedge \neg \mathsf{O}(X_i, \mathit{Stock})\big)\Big). \\
\Box_F^+ \big(\mathsf{P}(X_i, \mathit{Stock}) \to \Diamond_F \mathsf{P}(X_i, \mathit{Stock})\big).
\end{array}$$

Denote the resulting knowledge base by $\Sigma$. We can use it to answer queries like 'how much time the government needs to get rid of the disease' or 'when it is safe to buy new animals,' for instance, by checking whether formulas of the form

$$\begin{array}{l}
\bigcirc \cdots \bigcirc \mathsf{EQ}(\mathit{F\&M}, \bot). \\
\bigcirc \cdots \bigcirc \big(\neg \Diamond_F \mathsf{P}(X_i, \mathit{F\&M})\big)
\end{array}$$

are logical consequences of $\Sigma$.

It is worth noting that in this example we have a typical mixture of 'a sort of' model checking and deduction: while the map of the country is simulated by taking all RCC-8 relations which hold true between farms, towns, forests, etc., knowledge about fluents like *F&M* and *Stock* is incomplete, since it depends on the future development. So to decide whether $\Sigma \models \varphi$ holds or not *proper deduction* (or *theorem proving*) is required (cf. [Halpern and Vardi, 1991]).

#### 4.1.4 Modal formalisms for spatio-temporal reasoning

We saw in Section 2.2.4 that BRCC-8 can be embedded into the bimodal logic $\mathsf{S4}_u$ (which yields decidability) and then into the one-variable fragment of classical first-order logic (which yields NP-completeness). Similarly, the constructed temporalizations of BRCC-8 can be translated into the language $\mathcal{PST}$, or *propositional spatio-temporal language*, that contains the temporal operators $\mathcal{S}$ and $\mathcal{U}$, and the 'spatial' operators of $\mathsf{S4}_u$ (i.e., $\boldsymbol{I}$, $\boxed{\forall}$ and their duals). The intended models of $\mathcal{PST}$, called *topological $\mathcal{PST}$-models*, are triples of the form $\mathfrak{N} = \langle \mathfrak{T}, \mathfrak{F}, \mathfrak{U} \rangle$, in which $\mathfrak{T} = \langle U, \mathbb{I} \rangle$ is a topological space, $\mathfrak{F} = \langle W, < \rangle$ a flow of time, and $\mathfrak{U}$ is a *valuation* associating with every propositional variable $p$ and every $w \in W$ a set $\mathfrak{U}(p, w) \subseteq U$. $\mathfrak{U}$ is then extended to arbitrary $\mathcal{PST}$-formulas in the following way:

- $\mathfrak{U}(\psi \wedge \chi, w) = \mathfrak{U}(\psi, w) \cap \mathfrak{U}(\chi, w)$;
- $\mathfrak{U}(\neg\psi, w) = U - \mathfrak{U}(\psi, w)$;
- $\mathfrak{U}(\boxed{\forall}\, \psi, w) = U$ if $\mathfrak{U}(\psi, w) = U$, and $\mathfrak{U}(\boxed{\forall}\, \psi, w) = \emptyset$ otherwise;
- $\mathfrak{U}(\boldsymbol{I}\psi, w) = \mathbb{I}\mathfrak{U}(\psi, w)$;
- $x \in \mathfrak{U}(\psi \mathcal{U} \chi, w)$ iff there is $v > w$ such that $x \in \mathfrak{U}(\chi, v)$ and $x \in \mathfrak{U}(\psi, u)$ for all $u \in (v, w)$;
- $x \in \mathfrak{U}(\psi \mathcal{S} \chi, w)$ iff there is $v < w$ such that $x \in \mathfrak{U}(\chi, v)$ and $x \in \mathfrak{U}(\psi, u)$ for all $u \in (v, w)$.

In particular,

- $\mathfrak{U}(\Diamond_F \psi, v) = \bigcup_{v > w} \mathfrak{U}(\psi, v), \quad \mathfrak{U}(\Box_F \psi, w) = \bigcap_{v > w} \mathfrak{U}(\psi, v).$

A $\mathcal{PST}$-formula $\varphi$ is *satisfied in* $\mathfrak{N}$ if $\mathfrak{U}(\varphi, w) \neq \emptyset$ for some $w \in W$.

Say that a topological $\mathcal{PST}$-model $\mathfrak{N} = \langle \mathfrak{T}, \mathfrak{F}, \mathfrak{U} \rangle$ *satisfies* **FSA** if, for every variable $p$, there are finitely many sets $U_1, \ldots, U_n \subseteq U$ such that

$$\{\mathfrak{U}(p, w) : w \in W\} = \{U_1, \ldots, U_n\}.$$

We can now extend the translation $\cdot^*$ from BRCC-8 into $\mathsf{S4}_u$, defined in Section 2.2.4, to a translation from $\mathcal{ST}_2$-formulas into the language of $\mathcal{PST}$. For temporal region terms we need two extra clauses:

$$(t_1 \,\mathcal{U}\, t_1)^* = \boldsymbol{CI}(t_1^* \,\mathcal{U}\, t_2^*),$$
$$(t_1 \,\mathcal{S}\, t_1)^* = \boldsymbol{CI}(t_1^* \,\mathcal{S}\, t_2^*).$$

Note that we then also have:

$$(\bigcirc t)^* = \bigcirc t^*, \quad (\Diamond_F t)^* = \boldsymbol{CI}\Diamond_F t^*, \quad (\Box_F t)^* = \boldsymbol{CI}\Box_F t^*.$$

For atomic $\mathcal{ST}_2$-formulas $P(t_1, t_2)$, the translation $P(t_1, t_2)^*$ is defined in precisely the same way as in Section 2.2.4. Suppose now that $\varphi$ is an arbitrary $\mathcal{ST}_2$-formula. Then $\varphi^*$ denotes the result of replacing all occurrences of atoms $P(t_1, t_2)$ in $\varphi$ with $(P(t_1, t_2))^*$. It should be clear from the definition that we have:

**Theorem 28.** *An $\mathcal{ST}_2$-formula $\varphi$ is satisfiable in a tt-model* (*with* **FSA**) *based on a flow of tiem $\mathfrak{F}$ iff $\varphi^*$ is satisfiable in a topological $\mathcal{PST}$-model* (*with* **FSA**) *based on $\mathfrak{F}$.*

Unfortunately, we can't conclude from this result that $\mathcal{ST}_2$ is decidable. It is a challenging open problem to find out whether satisfiability of $\mathcal{PST}$-formulas in arbitrary topological models, or even only in those based on Kripke spaces, is decidable.

Note that the $\mathcal{PST}$-formula $\Diamond_F \boldsymbol{C}p \leftrightarrow \boldsymbol{C}\Diamond_F p$ is valid in all $\mathcal{PST}$-models based on Kripke spaces, but not on arbitrary topological spaces, simply because there is an infinite sequence of closed sets in $\mathbb{R}$ the union of which is not closed. However, the two types of models turn out to be equivalent with respect to the *modal translations* of (a) $\mathcal{ST}_1$-formulas, and (b) $\mathcal{ST}_2$-formulas provided that models satisfy the *finite state assumption* **FSA**. Moreover, in both cases we can again take advantage of the special form of these translations and show that $\mathcal{PST}$-models based on quasisaw Kripke spaces are enough to satisfy all satisfiable formulas (see [Gabbay *et al.*, 2001] for a proof):

**Theorem 29.** (i) *An $\mathcal{ST}_2$-formula $\varphi$ is satisfied in a tt-model with* **FSA** *based on a flow of time $\mathfrak{F}$ iff $\varphi^*$ is satisfied in a $\mathcal{PST}$-model with* **FSA** *based on $\mathfrak{F}$ and a quasisaw Kripke space.*

(ii) *An $\mathcal{ST}_1$-formula $\varphi$ is satisfied in a tt-model based on a flow of time $\mathfrak{F}$ iff $\varphi^*$ is satisfied in a $\mathcal{PST}$-model based on $\mathfrak{F}$ and a quasisaw Kripke space.*

We can now use this result to 'lift' the translation $\cdot^\dagger$ of BRCC-8 formulas into the one-variable fragment of first-order logic to a translation of $\mathcal{ST}_2$-formulas into the one-variable fragment of first-order temporal logic. This can be done by adding to the definition of the translations $\cdot^b$, $\cdot^l$, and $\cdot^r$ in Section 2.2.4 two more clauses:

$$(\varphi\, \mathcal{U}\, \psi)^i = \varphi^i\, \mathcal{U}\, \psi^i, \quad \text{for } i = b, l, r,$$
$$(\varphi\, \mathcal{S}\, \psi)^i = \varphi^i\, \mathcal{S}\, \psi^i, \quad \text{for } i = b, l, r.$$

Given an $\mathcal{ST}_2$-formula $\varphi$, we put $\varphi^\dagger = (\varphi^*)^b$. Note that, as before, $\varphi^\dagger$ contains a single individual variable.

**Theorem 30.** *Suppose $\mathfrak{F}$ is a flow of time and $\varphi$ an $\mathcal{ST}_2$-formula. Then the following conditions are equivalent:*

1. *$\varphi^*$ is satisfiable in a $\mathcal{PST}$-model (with* **FSA***) based on $\mathfrak{F}$ and a quasisaw Kripke space;*

2. *$\varphi^\dagger$ is satisfiable in a first-order temporal model based on $\mathfrak{F}$ (and having a finite domain).*

Now a proof of Theorem 26 (i) is obtained by combining Theorem 20 (i) and (iii) with Theorem 29 (ii) and Theorem 30. A proof of Theorem 27 (i) follows from Theorem 20 (ii), (iii), and Theorems 29 and 30.

#### 4.1.5 Temporal models based on Euclidean spaces

As we observed in Section 2.2.2, there exist satisfiable $\mathsf{BRCC\text{-}8}$ formulas that are not satisfiable in any connected (in particular, Euclidean) topological space. A simple example is the conjunction $\varphi$ of the following predicates:

$$\mathsf{EQ}(X_1 \sqcup X_2, Y), \quad \mathsf{NTPP}(X_1, Y), \quad \mathsf{NTPP}(X_2, Y), \quad \mathsf{DC}(Y, Z).$$

Clearly, $\varphi$ is satisfied in the discrete space with three points. Note now that if $\varphi$ holds in some topological space, then $X_1 \sqcup X_2$ is closed and included in the interior of $Y$. On the other hand, it coincides with $Y$. Hence, $Y$ is both closed and open. However, $Y$ is not the whole space because it is disjoint with $Z$.

A similar effect can be achieved in the spatio-temporal case even without using the Boolean operations on region terms simply because unions of regions are implicitly available in $\mathcal{ST}_2$ in the form of $\Diamond_F$. Consider, for instance, the conjunction $\psi$ of the predicates:

$$\mathsf{EQ}(\Diamond_F X, Y), \quad \mathsf{NTPP}(\bigcirc X, Y), \quad \mathsf{NTPP}(\bigcirc \Diamond_F X, Y), \quad \mathsf{DC}(Y, Z).$$

One can readily check that $\psi$ is satisfiable in some tt-model with **FSA**, but not in a model based on a connected topological space, in particular $\mathbb{R}^n$, for any $n \geq 1$.

It is an interesting open problem whether satisfiability of $\mathcal{ST}_2$-formulas (with or without the Booleans on region terms) in models (with **FSA**) based on Euclidean spaces is decidable (cf. [Renz, 1998]). We only know that the following holds (see [Wolter and Zakharyaschev, 2000b] for a proof):

**Theorem 31.** *If a set of $\mathcal{ST}_1$-formulas without Boolean operations on region terms is satisfiable in a tt-model based on $\langle \mathbb{N}, < \rangle$, then it is also satisfiable in a model based on $\langle \mathbb{N}, < \rangle$ and $\mathbb{R}^n$, for any $n \geq 1$.*

### 4.2 Spatio-temporal logics of branching time

In the framework of linear time spatio-temporal logics, we can say, for instance, that the UK will join the euro-zone: $\Diamond_F \mathsf{P}(\mathit{UK}, \mathit{Euro\text{-}zone})$. We can also say that this will never happen. But we are not able to convey the reality, viz., that both variants are possible:

$$\Diamond \Diamond_F \mathsf{P}(\mathit{UK}, \mathit{Euro\text{-}zone}) \land \Diamond \neg \Box_F \mathsf{P}(\mathit{UK}, \mathit{Euro\text{-}zone}).$$

Nor can we make the foot and mouth scenario above more realistic by saying that the disease *possibly* spreads to the neighboring farms. In this section, we show how the spatio-temporal formalisms developed so far can be extended to the branching time paradigm capable of making assertions about alternative histories.

At the syntactical level we have two options: to allow applications of $\Box$ and $\Diamond$ only to $\mathcal{ST}_i$-formulas, or to both formulas and temporal region terms. The resulting languages will be denoted by $\mathcal{STB}_i$ (the former option) and $\mathcal{STB}_i^+$ (the latter one). In the latter case, we also have to update the notion of temporal region term by adding to its definition the clause: if $t$ is a temporal region term, then so are $\Box t$ and $\Diamond t$. For example, the following $\mathcal{STB}_2^+$-formula

$$\Box\Box_F\big(\mathsf{EQ}(\mathit{Europe}, \bigcirc \mathit{Europe}) \wedge \mathsf{P}(\mathit{EU}, \mathit{Europe})\big) \wedge$$
$$\mathsf{P}(\mathit{Europe}, \Diamond \bigcirc \mathit{EU}) \wedge \mathsf{P}(\Box \bigcirc \mathit{EU}, \mathit{EU})$$

says that, whatever happens, the region occupied by Europe will always remain the same and the EU will be part of Europe; moreover, every part of Europe has a possibility to join the EU next year, while, on the hand, what will certainly belong to the EU next year, is only part of the EU as it is today.

The extension of tt-models to branching time topological models is straightforward:

**Definition 32 (branching tt-models).** A *branching time topological model* (a *btt-model*, for short) is a triple $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{F}, \mathcal{H}\rangle$, where $\mathfrak{S}$ is a topological model, $\mathfrak{F} = \langle W, <\rangle$ a tree, and $\mathcal{H}$ a set of histories in $\mathfrak{F}$. An *assignment* $\mathfrak{a}$ in $\mathfrak{M}$ associates with every region variable $X$ and every $w \in W$ a set $\mathfrak{a}(X, w) \in \mathcal{R}(\mathfrak{T})$.

Given a region term $t$, a time point $w \in W$, and a history $h \in \mathcal{H}$, define the *value* $\mathfrak{a}(t, h, w)$ of $t$ at $w$ relative to $h$ inductively by taking

$$\begin{aligned}
\mathfrak{a}(X, h, w) &= \mathfrak{a}(X, w), \ \ X \text{ a region variable;}\\
\mathfrak{a}(\Diamond t, h, w) &= \mathbb{C}\mathbb{I} \bigcup_{h' \in \mathcal{H}(w)} \mathfrak{a}(t, h', w);\\
\mathfrak{a}(\Box t, h, w) &= \mathbb{C}\mathbb{I} \bigcap_{h' \in \mathcal{H}(w)} \mathfrak{a}(t, h', w);\\
\mathfrak{a}(t \mathcal{U} s, h, w) &= \mathbb{C}\mathbb{I}\, \{x \in U : \exists v > w \ (v \in h \wedge x \in \mathfrak{a}(t, h, v) \ \wedge\\
&\qquad \forall u \in (w, v) \ x \in \mathfrak{a}(s, h, u)\},
\end{aligned}$$

the standard clauses for the Booleans and a dual clause for $\mathcal{S}$. Now, for a formula $\varphi$ and a pair $(h, w)$, the *truth* of $\varphi$ at $(h, w)$ in $\mathfrak{M}$ is defined inductively as follows:

- $(h, w) \models^{\mathfrak{a}} P(s, t)$ iff $\mathfrak{S} \models P[\mathfrak{a}(s, h, w), \mathfrak{a}(t, h, w)]$, for atomic $P(s, t)$;
- $(h, w) \models^{\mathfrak{a}} \psi \mathcal{U} \chi$ iff there is $v > w$ such that $v \in h$, $(h, v) \models^{\mathfrak{a}} \chi$, and $(h, u) \models^{\mathfrak{a}} \psi$ for all $u \in (w, v)$;
- $(h, w) \models^{\mathfrak{a}} \Diamond\varphi$ iff there is $h' \in \mathcal{H}(w)$ such that $(h', w) \models^{\mathfrak{a}} \varphi$;
- $(h, w) \models^{\mathfrak{a}} \Box\varphi$ iff $(h', w) \models^{\mathfrak{a}} \varphi$ for all $h' \in \mathcal{H}(w)$,

plus the standard clauses for the Booleans and $\mathcal{S}$.

The computational behavior of the spatio-temporal logics of branching time is similar to that we have observed above in the linear case. First, we have:

**Theorem 33.** *There is an algorithm which, given an $\mathcal{STB}_1$-formula $\varphi$, decides whether $\varphi$ is satisfiable in a btt-model or not.*

This result can be proved by extending the embedding of $\mathcal{ST}_1$ into $\mathcal{FOTL}$ to an embedding of $\mathcal{STB}_1$ into the one-variable fragment of $\mathcal{FOBTL}$ and then applying Theorem 21 (i). No significant result on the computational complexity of $\mathcal{STB}_1$-formulas has been obtained yet.

As to satisfiability of $\mathcal{STB}_i^+$-formulas, we again face the problem of infinitary operations on temporal spatial terms. Now, besides the temporal operators, the spatial terms can also be affected by the modal operators $\Box$ and $\Diamond$. Say that a btt-model $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{F}, \mathcal{H} \rangle$ is a *finite branching model* if the set $\mathcal{H}$ of histories in it is finite. The *finite state assumption* **FSA** is applied now to each history.

**Theorem 34.** (i) *It is decidable whether an $\mathcal{STB}_1^+$-formula is satisfiable in a finite branching model.*

(ii) *It is decidable whether an $\mathcal{STB}_2^+$-formula is satisfiable in a finite branching model with* **FSA**.

The proof is conducted by embedding $\mathcal{STB}_2^+$ into the one-variable fragment of $\mathcal{FOBTL}$ and applying Theorem 21 (ii). Nothing is known about the computational complexity of these satisfiability problems yet. The decidability of the satisfiability problem for $\mathcal{STB}_2^+$-formulas in arbitrary btt-models or only in those based on Euclidean spaces is also open.

## 4.3 BRCC-8 + $\mathcal{A}\ell\ell$-13

Since the region-based approach to spatial reasoning was inspired by and closely mirrors the interval-based approach to temporal reasoning—they both take extended entities rather than points as primitives—it would seem far more natural to temporalize BRCC-8 by combining it with an interval based temporal logic. In this section we show a variant of such a combination.

Following [Allen, 1984] we write $\mathsf{HOLDS}(\varphi, i)$ to say that a formula $\varphi$ holds during a time interval $i$. For example, $\mathsf{HOLDS}\big(\mathsf{PO}(X,Y), i\big)$ means that during interval $i$ regions $X$ and $Y$ partially overlap. Let us call an $\mathcal{ARCC}$-8 *formula* any Boolean combination of atomic $\mathcal{A}\ell\ell$-13 formulas, and formulas of the form $\mathsf{HOLDS}(\varphi, i)$, where $\varphi$ is a BRCC-8 formula.

$\mathcal{ARCC}$-8 formulas are interpreted in standard topological temporal models $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{F} \rangle$ based on linear flows of time. The only essential difference is that now an *assignment* $\mathfrak{a}$ in $\mathfrak{M}$ associates with every interval variable $i$ a non-empty convex set $\mathfrak{a}(i)$ in $\mathfrak{F}$, and with every region variable $X$ and every time point $w$ it associates a regular closed set $\mathfrak{a}(X, w)$ in $\mathfrak{S}$. The truth-relation for the $\mathcal{A}\ell\ell$-13 atomic formulas is defined as in Section 3.4, and $\mathsf{HOLDS}(\varphi, i)$ is true in $\mathfrak{M}$ iff for every point $w \in \mathfrak{a}(i)$, we have $\mathfrak{S} \models^{\mathfrak{a}_w} \varphi$ (as defined in Sections 2.2.1 and 2.2.2).

Here is a simple example of a 'knowledge base' $\Sigma$ in this unsophisticated language:

$$\mathsf{meets}(i,j) \wedge \mathsf{during}(i,k) \wedge \mathsf{during}(j,k).$$
$$\mathsf{HOLDS}\big(\mathsf{TPP}(\mathit{Hong\_Kong}, \mathit{UK}) \wedge \mathsf{EC}(\mathit{Hong\_Kong}, \mathit{China}), i\big).$$
$$\mathsf{HOLDS}\big(\mathsf{DC}(\mathit{Hong\_Kong}, \mathit{UK}), j\big).$$
$$\mathsf{HOLDS}\big(\mathsf{EC}(\mathit{UK}, \mathit{China}) \vee \mathsf{DC}(\mathit{UK}, \mathit{China}), k\big).$$

If $\Sigma$ is true in a tt-model $\mathfrak{M}$ under an assignment $\mathfrak{a}$, then the formula $\varphi$

$$\mathsf{HOLDS}\big(\mathsf{EC}(\mathit{UK}, \mathit{China}), i\big)$$

also holds in $\mathfrak{M}$ under $\mathfrak{a}$, i.e., $\varphi$ is a logical consequence of $\Sigma$.

A straightforward combination of the satisfiability checking algorithms for $\mathcal{A}\ell\ell$-13 and BRCC-8 yields a satisfiability-checking algorithm for $\mathcal{ARCC}$-8. More precisely, we have the following:

**Theorem 35.** *The satisfiability problem for $\mathcal{ARCC}$-8 formulas in tt-models is NP-complete.*

An interesting open problem is to find *tractable* fragments of $\mathcal{ARCC}$-8, for instance, by combining tractable fragments of RCC-8 and $\mathcal{A}\ell\ell$-13 [Nebel and Bürckert, 1995; Renz and Nebel, 1999].

As was noted in Section 3.4, $\mathcal{A}\ell\ell$-13 can be embedded into propositional temporal logic. Together with the modal translation of BRCC-8, this yields an embedding of $\mathcal{ARCC}$-8 into the language $\mathcal{PST}$ interpreted in topological $\mathcal{PST}$-models based on linear flows of time, and then into the one-variable fragment of first-order temporal logic. For details the reader is referred to [Bennett *et al.*, 2001; Gabbay *et al.*, 2001].

# 5 Concluding remarks

Now, as we have constructed a family of decidable spatio-temporal formalisms, a natural question is whether they are 'implementable.'

## 5.1 Implementable algorithms

The satisfiability problems for all our logics are polynomially reducible to the satisfiability problems for the one-variable fragments of first-order temporal logics. We have also seen that usually these fragments are decidable. So the question is whether they can be supported by 'practical' decision procedures.

One idea would be take advantage of the fact that the one-variable (and other monodic) fragments of many temporal logics are embeddable into monadic second-order logic (in the case of **FCA** or **FSA** even weak monadic second-order logic may be enough) [Hodkinson *et al.*, 2000] and use provers like MONA (see e.g. [Klarlund *et al.*, 2000] and references therein). Unfortunately, however, the translation from [Hodkinson *et al.*, 2000] is exponential, which will make the prover's job much harder.

On the other hand, a tableau decision procedure for the one-variable fragment of first-order temporal logic based on $\langle \mathbb{N}, < \rangle$ has been developed in [Lutz *et al.*, 2001] as a combination of Wolper's [1985] tableau for $\mathcal{PTL}$ and a standard tableau for (the one-variable fragment of) first-order logic. Currently, a similar tableau-based procedure is being implemented for temporal description logic [Günsel and Wittmann, 2001], and we expect significant experimental results on the efficiency of the procedure shortly. Positive results would allow the construction of a practical system for the language $\mathcal{ST}_1$ (without $\mathcal{S}$) interpreted in tt-models based on $\langle \mathbb{N}, < \rangle$ and possibly infinite topological spaces. That would also open the door to an implementation of a decision procedure for $\mathcal{ST}_2$ in models based on $\langle \mathbb{N}, < \rangle$ and *finite* topological spaces.

## 5.2 Further extensions

The obtained results make only first steps in the study of effective spatio-temporal formalisms. Many interesting problems remain open for investigation. For instance, it would be interesting and practically important to extend the spatio-temporal logics with constructors allowing us to speak about orientation (say, go West), change, and distances. On the other hand, we also need constructors to represent properties of regions different from purely spatial or temporal (e.g. 'region $X$ is a tourist attraction accessible only by plane'). A promising idea is to combine spatio-temporal logics with suitable description logics (see e.g. [Haarslev *et al.*, 1998]; a combination of description logic with metric logic has been proposed in [Kutz *et al.*, 2001].)

An interesting problem is to find and temporalize more expressive and still decidable fragments of RCC. For example, one can consider a Datalog-type language with built-in basic spatial predicates. We could then compose knowledge bases like:

$$\mathsf{P}(Y, \textit{euro-zone}) \leftarrow \mathsf{P}(X, \textit{euro-zone}) \wedge \mathsf{EC}(X, Y) \wedge \mathsf{PP}(Y, \textit{europe}).$$
$$\mathsf{P}(\textit{germany,euro-zone}).$$
$$\mathsf{EC}(\textit{germany,poland}).$$
$$\mathsf{PP}(\textit{poland,europe}).$$
$$\mathsf{EC}(\textit{poland,russia}).$$
$$\mathsf{O}(\textit{russia,asia}).$$
$$\mathsf{EC}(\textit{europe,asia}).$$

The answer to the query ?$\mathsf{P}$(*poland,euro-zone*) should be YES, while the answer to ?$\mathsf{P}$(*russia,euro-zone*) should be NO. The language can be extended with Boolean region terms.

Till now we have not imposed any restrictions on the form of spatial regions. However, applications in GIS may require to consider only the Euclidean space $\mathbb{R}^2$ and interpret regions in them as figures of some special form, say, as circles or polygons [Grigni *et al.*, 1995]. But it is still an open problem whether the satisfiability problem for RCC-8 formulas under such interpretations is decidable. A similar question can be asked regarding BRCC-8 and the spatio-temporal logics constructed above.

Halpern and Shoham [1991] introduced a modal logic of intervals whose modal operators correspond to the thirteen relations of Allen's interval logic (see Section 3.4). One can construct a similar modal logic[9] of regions with eight modal operators of the form $\langle \mathsf{TPP} \rangle$. The intended meaning of these modalities is as follows. Suppose we have a topological model $\mathfrak{S}$ of the form (4) and a region $X$ in $\mathfrak{T}$. Then $\langle \mathsf{TPP} \rangle \varphi$ holds at $X$ in $\mathfrak{S}$ if there is a region $Y$ in $\mathfrak{T}$ such that $\mathsf{TPP}(X, Y)$ and $\varphi$ holds at $Y$. An interesting research problem is to investigate the computational behavior of this logic for different classes of topological models. (The logic of Halpern and Shoham is undecidable if intervals are taken on an infinite time-line.)

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[9]The idea is due to C. Lutz.

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