Mereology 1

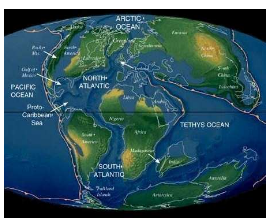
Thomas Bittner
thomas.bittner@ifomis.uni-leipzig.de

Overview

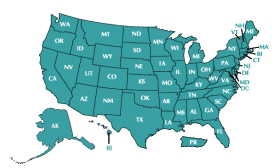
- Introduction and examples
- Mereology as formal theory
- Ground mereology (GM) as partial ordering
- Models of ground mereology (GM)
- Some defined primitives of (GM)
- Consequences of axioms and definitions
- Some theorems of (GM)
- Assignments and Summary

The intuitive meaning of the word part-of and examples of its linguistic use

Parts of the earth

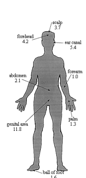


Federal States of the USA

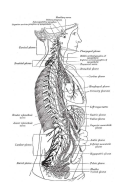


Physical geography of the USA

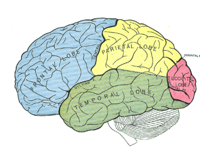
Some parts of the human body



Parts of the nerve system



Parts of the brain



Usage of the word part in natural language

1. The handle is part of the cup.
2. This cap is part of my pen.
3. The left half is your part of the cake.
4. The US is part of North America.
5. The contents of this bag is only part of what I bought.
6. That corner is part of the living room.
7. The outermost points are part of the perimeter.
8. The first act was the best part of the play.

used to indicate any portion of a given entity that is

- attached to the remainder
  - The handle is part of the cup.
- detached from the remainder:
  - This cap is part of my pen.
- arbitrarily demarcated,
  - The left half is your part of the cake.

used to indicate any portion of a given entity that is

- self-connected
  - The handle is part of the cup.
  - The left half is your part of the cake.
- disconnected
  - The US is part of North America.
- homogeneous
  - The handle is part of the cup.
  - The US is part of North America.

used to indicate any portion of a given entity that is

- Gerrymandered
  - The contents of this bag is only part of what I bought.
- Material
  - The handle is part of the cup.
  - The contents of this bag is only part of what I bought.
- Immaterial
  - That corner is part of the living room.

used to indicate any portion of a given entity that is

- Extended
  - The handle is part of the cup.
  - That corner is part of the living room.
- Unextended
  - The outermost points are part of the perimeter.

used to indicate any portion of a given entity that is

- Spatial
  - The handle is part of the cup.
  - The outermost points are part of the perimeter.
- Temporal
  - The first act was the best part of the play.

Abstract parts

- The integers are part of the reals
- The first chapter is part of the novel.
- Humanity is part of personhood

Non-mereological use of the word part

- The clay is part of the statue
  - Constitution
- The gin is part of martini
  - Chemical composition
- Writing comments is part of being a good referee
  - Conceptual inclusion

Mereology as a formal theory of parthood

Formal axiomatic theories of mereology

- Formulated in standard first or second order logic
- Include special symbols which are supposed to designate mereological relations or functions
- Some relations are treated as *primitives*: they are not defined. Axioms stipulating their logical properties are included the theory
- Other relations are *defined in terms of the primitives*

Formal axiomatic theories of mereology (2)

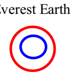
- A consistent formal theory has *models*:
  - models are collections of individuals
  - which satisfy the axioms of the theory when the primitives are interpreted in a certain way with respect to these individuals.

The logical language

- First order language with identity
- Variables ranging over individuals, x,y,z, …
- Logical connectivities: &, or, ⇒, ⇔
- Universial and extensional quantification: (x), (∃x)
- Leading universal quantifiers are omitted, i.e., we write P xx instead of (x)(P xx)

The non-logical primitive

- P xy
- Intended interpretation: x is a part of y
- examples
  - P your-hand you
  - P MountEverest Earth
  - P xy



Other authors have used other relations as primitives:

- Proper parthood (Simons)
- Disjointness (Leonard and Goodmann)
- Overlap (???)

Axioms

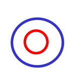
- Specify the meaning of the non-logical primitives
- Constrain the models of the theory
- Mereological structures
  - Models of mereology
  - Special kind of partial orderings

Parthood as partial ordering (1)

- Axiom of reflexivity
  - Everything is part of itself
  - P xx
- Axiom of antisymmetry
  - If two things are parts of each other then they are identical
  - P xy & P yx ⇒ x = y

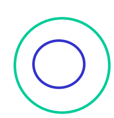
Parthood as partial ordering (2)

- Axiom of transitivity
  - P xy



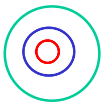
Parthood as partial ordering (2)

- Axiom of transitivity
  - P xy & P yz



Parthood as partial ordering (2)

- Axiom of transitivity
  - P xy & P yz ⇒ P xz



Axioms of Ground mereology - **M**

- M1 P xx
- M2 P xy & P yx ⇒ x = y
- M3 P xy & P yz ⇒ P xz

Models of Ground Mereology

Algebraic structures

- An algebraic structure is a pair (S,R)
  - S is a set of entities
  - R is a binary relation
- Examples
  - $A_1 = (S_1,R_1)$
    - S_1 = {a,b}
    - R_1 = {(a,a), (a,b), (b,b) }
  - $A_2 = (S_2,R_2)$
    - S_2 = {a,b,c}
    - R_2 = {(a,a), (a,b), (b,b) ,(b,c),(cc)}

Models

- An algebraic structure *A=(S,R)* is a model of a set of axioms with a single non-logical primitive *P* if and only if
  - The non-logical primitive *P* is interpreted as the the relation *R* of the structure
  - The axioms are true for each assignment of the variables with entities of the domain *S*

Example 1

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- We interpret P as R_1
- We then need to verify that the axioms
  - P xx
  - P xy & P yx ⇒ x = y
  - P xy & P yz ⇒ P xz
- For any assignment of entities in S_1 to the variables

Example 1 (cont.)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- P xx

x	Pxx
a	T

Example 1 (cont.)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- P xx

x	Pxx
a	T
b	T

Example 1 (cont.)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
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- P xx

x	Pxx
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Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- P xy & P yx ⇒ x = y

x	y	P xy	P yx	⇒	x = y
a	a	T	T	T	T

Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- P xy & P yx ⇒ x = y

x	y	P xy	P yx	⇒	x = y
a	a	T	T	T	T
a	b	T	F	T	F

Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- P xy & P yx ⇒ x = y

x	y	P xy	P yx	⇒	x = y
a	a	T	T	T	T
a	b	T	F	T	F
b	a	F	T	T	F

Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- P xy & P yx ⇒ x = y

x	y	P xy	P yx	⇒	x = y
a	a	T	T	T	T
a	b	T	F	T	F
b	a	F	T	T	F
b	b	T	T	T	T

Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- P xy & P yx ⇒ x = y

x	y	P xy	P yx	⇒	x = y
a	a	T	T	T	T
a	b	T	F	T	F
b	a	F	T	T	F
b	b	T	T	T	T

Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- P xy & P yz ⇒ P xz

x	y	z	P xy	P yz	⇒	P xz
a	a	b	T	T	T	T

Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- P xy & P yz ⇒ P xz

x	y	z	P xy	P yz	⇒	P xz
a	a	b	T	T	T	T
a	b	a	T	F	T	T

Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
  - R_1 = {(a,a), (a,b), (b,b) }
- P xy & P yz ⇒ P xz

x	y	z	P xy	P yz	⇒	P xz
a	a	b	T	T	T	T
a	b	a	T	F	T	T
b	a	b	F	T	T	T

Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - S_1 = {a,b}
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- P xy & P yz ⇒ P xz

x	y	z	P xy	P yz	⇒	P xz
a	a	b	T	T	T	T
a	b	a	T	F	T	T
b	a	b	F	T	T	T
b	b	a	T	F	T	F
…						

Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - $S_1 = \{a,b\}$
  - $R_1 = \{(a,a), (a,b), (b,b)\}$
- P xy & P yz ⇒ P xz

x	y	z	P xy	P yz	⇒	P xz
a	a	b	T	T	T	T
a	b	a	T	F	T	T
b	a	b	F	T	T	T
b	b	a	T	F	T	F
…						

Example 1 (cont)

- Structure $A_1 = (S_1,R_1)$ with
  - $S_1 = \{a,b\}$
  - $R_1 = \{(a,a), (a,b), (b,b)\}$

is a model of the axioms of ground mereology

- M1 P xx
- M2 P xy & P yx ⇒ x = y
- M3 P xy & P yz ⇒ P xz

Example 2

- Structure $A_2 = (S_2,R_2)$ with
  - $S_2 = \{a,b,c\}$
  - $R_2 = \{(a,a), (a,b), (b,b), (b,c), (cc)\}$
- We interpret P as R_2
- We then need to verify that the axioms
  - P xx
  - P xy & P yx ⇒ x = y
  - P xy & P yz ⇒ P xz
- For any assignment of entities in S_2 to the variables

Example 2

- Structure $A_2 = (S_2,R_2)$ with
  - $S_2 = \{a,b,c\}$
  - $R_2 = \{(a,a), (a,b), (b,b), (b,c), (cc)\}$
- P xy & P yz ⇒ P xz

Example 2 (cont.)

- Structure $A_2 = (S_2,R_2)$ with
  - $S_2 = \{a,b,c\}$
  - $R_2 = \{(a,a), (a,b), (b,b), (b,c), (cc)\}$
- P xy & P yz ⇒ P xz

x	y	z	P xy	P yz	⇒	P xz
…						
a	b	c	T	T	F	F
…						

Example 2 (cont.)

- Structure $A_2 = (S_2,R_2)$ with
  - $S_2 = \{a,b,c\}$
  - $R_2 = \{(a,a), (a,b), (b,b), (b,c), (cc)\}$
- P xy & P yz ⇒ P xz

x	y	z	P xy	P yz	⇒	P xz
…						
a	b	c	T	T	F	F
…						

- A_2 is NOT a model of ground mereology

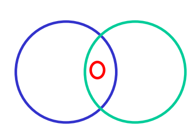
Some defined primitives of ground mereology

Ground mereology - **M**

- Axioms
  - M1 P xx
  - M2 P xy & P yx ⇒ x = y
  - M3 P xy & P yz ⇒ P xz
- Defined relations:
  - Overlap
  - Underlap
  - Proper part

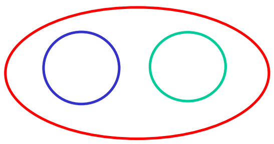
Definition of overlap

D_O: O xy iff (∃z)(P zx & P zy)



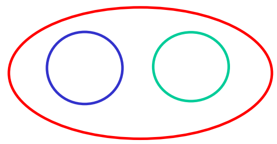
Definition of underlap

D_U: U xy iff $(\exists z)$(P xz & P yz)



Definition of proper part

D_{PP}: PP xy iff P xy & $\neg$P yx



Remainder: Formal axiomatic theories of mereology

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Remainder: Formal axiomatic theories of mereology (2)

- A consistent formal theory has *models*:
  - models are collections of individuals
  - which satisfy the axioms of the theory when the primitives are interpreted in a certain way with respect to these individuals.

Exploring the logical consequences of our axioms and definitions

- Proving theorems using the underlying logical calculus

Axioms definitions —Rules of inference→ Theorems

Rules of inference (Copi 1979)

Modus Ponens (MP)

$p \Rightarrow q$,
p
$\therefore q$

Example:

…
2 P xy & P yx $\Rightarrow$ x=y
3 P xy & P yx
4 x=y 2,3 MP
…

Rules of inference (Copi 1979)

- Modus Tollens (MT)

$p \Rightarrow q$
$\neg q$
$\therefore \neg p$

…
2 P xy & P yx $\Rightarrow$ x=y
3 $\neg$ (x=y)
4 $\neg$ (P xy & P yx) 2,3 MP
…

Rules of inference (Copi 1979)

- Hypothetical Syllogism (HS)

$p \Rightarrow q$
$q \Rightarrow r$
$\therefore p \Rightarrow r$

…
2 P xy & P yz $\Rightarrow$ P xz
3 P zx $\Rightarrow$ O xz
4 P xy & P yz $\Rightarrow$ O xz 2,3 HS
…

Rules of inference (Copi 1979)

- Disjunctive Syllogism (DS)

p or q
$\neg p$
$\therefore q$

…
2 P xy or P yz
3 $\neg$ P xy
4 P yz 2,3 DS
…

Rules of inference (Copi 1979)

- Constructive Dilemma (CD)
  - (p ⇒ q) & (r ⇒ s), p or r ∴ q or s
- Destructive Dilemma (DD)
  - (p ⇒ q) & (r ⇒ s), ¬q or ¬s ∴ ¬p or ¬r
- Simplification (Simp)
  - p & q ∴ p
- Conjunction (conj)
  - p, q ∴ p & q
- Addition (add)
  - p ∴ p or q

Rules of inference (Copi 1979)

- Rule of replacement:
  - Any logically equivalent expressions can be replaced by each other whenever they occur
  - ¬(p & q) ⇔ ¬p or ¬q DeM
  - ¬(p or y) ⇔ ¬p & ¬q DeM
  - (p or q) ⇔ (q or p) comm
  - (p & q) ⇔ (q & p) comm
  - (p & (q & r)) ⇔ (p & q) & r asso
  - …
  - p ⇔ ¬ ¬p DN

Rule of replacement: example

…
3 P xy & ¬(P yz & P zu)
4 P xy & (¬P yz or ¬ P zu) 3 DeM
5 (P xy & ¬P yz) or (P xy & ¬ P zu) 4 Dist
…

Rules of inference (Copi 1979)

- Rule of conditional proof:

1	p	ass
|		
|		
n	q	
n+1	q ⇒ q	1-n CP

Example Condional Proof

M |-- PP xy ⇒ P xy

1	PP xy	ass
2	P xy & ¬P yx	1 D_{PP}
3	P xy	3 simp
4	PP xy ⇒ P xy	1-3 CP

Rules of inference (Copi 1979)

- Indirect proofs

p	¬PP xx	
…	1. PP xx	ass
…	2. P xx & ¬P xx	1 D_{PP}
q & ¬ q	3. ¬PP xx	1-2 I
∴ ¬ p		

Quantification rules (Copi 1979)

- Universal instantiation
  - (x)(P xy)
    - P xy UI
    - P yy UI
    - P ay UI
  - (x) (P xy ⇒ (∃z)(P zx))
    - P xy ⇒ (∃z)(P zx) UI
    - P zy ⇒ (∃z)(P zz) WRONG

Example

…
3 ¬P zz or P xy
4 (x) Pxx M1
5 P zz 4 UI
6 P xy 3,5 DS
…

Quantification rules (Copi 1979)

- Existential generalization
  - P xy
    - (∃x)(P xy) EG
    - (∃y)(P xy) EG
  - P ay
    - (∃x)(P xy) EG
  - Pyy
    - (∃x)(P xy) EG

Example

…

4	P zx & P zy	
5	(∃z)(P zx & P zy)	4 EG
6	O xy	5 D_O

…

Quantification rules (Copi 1979)

- Existential instantiation
  - (∃x)(P xy)
    - P xy — if x has not occurred free
    - P ay — if a is a new constant

1 (∃x)(P xy)
2 P xy
…
n P zy
∴ P zy 1-n EI

Example proof

M |-- O xy ⇒ O yx

1.	O xy	ass
2.	(∃z)(P zx & P zy)	1 D_O
3.	P zx & P zy	
4.	P zy & P zx	3 assoc
5.	(∃z)(P zy & P zx)	4 EG
6.	(∃z)(P zy & P zx)	3-5 EI
7.	O yx	6 D_O
8.	O xy ⇒ O yx	1-7 CP

Quantification rules (Copi 1979)

- Universal generalization

…
P xy
(x) Pxy UG

  - Except if x occurs free in some open assumption:

1	P xy	ass
…		
n	P xz	
n+1	(x) P xz	WRONG
n+2	P xy ⇒ (x) P xz	1-n+2 CP

More equivalences and rules

- Quantifier negation
  - (x) ¬ P xy ⇔ ¬(∃x)P xy
  - ¬(x) P xy ⇔ (∃x) ¬P xy
  - …

…

3	¬O xy	ass
4	¬(∃z)(P zx & P zy)	3 D_O
5	(z) ¬(P zx & P zy)	4 QN
6	(z) (¬P zx or ¬ P zy)	5 DeM
7	¬P xx or ¬ Pxy	6 UI
8	¬ Pxy	7,M1 DS
9	¬O xy ⇒ ¬ Pxy	3-8 CP

…

Theorems of Ground Mereology

- M |-- O xx
- M |-- O xy ⇒ O yx
- …
- M |-- P xy ⇔ (PP xy or x=y)
- M |-- P xy & ¬ (x=y) ⇒ PP xy
- …

Some example proofs

M |-- O xx

1.	P xx	M1
2.	P xx & P xx	1 taut
3.	(∃z)(P zx & P zx)	2 EG
4.	O xx	3 D_O

P xy & ¬ (x=y) ⇒ PP xy

1	P xy & ¬ (x=y)	ass
2	¬ (x=y)	1 simp
3	P xy & P yx ⇒ x = y	M2 UI
4	¬ (P xy & P yx)	2,3 MT
5	¬ P xy or ¬ P yx	4 DeM
6	P xy	1 simp
7	¬ P yx	5,6 DS
8	P xy & ¬ P yx	6,7 conj
9	PP xy	8 D_{PP}
10	P xy & ¬ (x=y) ⇒ PP xy	1-9 CP

P xy ⇒ (PP xy or x=y)

1	P xy	ass
2	¬(PP xy or x=y)	ass
3	¬PP xy & ¬ x=y	2 DeM
4	¬ x=y	3 simp
5	P xy & P yx ⇒ x = y	P2 UI
6	¬(P xy & P yx)	5,4 MT
7	¬P xy or ¬ P yx	6 DeM
8	¬ P yx	7,1 DS
9	P xy & ¬ P yx	1,8 conj
10	PP xy	9 D_{PP}
11	PP xy & ¬PP xy	10,(3 simp) conj
12	(PP xy or x=y)	2-11 IP
13	P xy ⇒ (PP xy or x=y)	1-12 CP

(PP xy or x=y) ⇒ P xy

1.	PP xy or x=y	ass
2.	PP xy	ass
3.	P xy & ¬ P yx	2 D_{PP}
4.	P xy	3 simp
5.	PP xy ⇒ P xy	2-4 CP
6.	x = y	ass
7.	P xx	M1 UI
8.	P xy	6,7 Id
9.	x=y ⇒ P xy	6-8 CP
10.	(PP xy ⇒ P xy) & (x=y ⇒ P xy)	5,9 conj
11.	P xy or P xy	10, 1 CD
12.	P xy	11 taut
13.	(PP xy or x=y) ⇒ P xy	1-12 CP

PP xy ⇒ ¬ PP yx

1	PP xy	ass
2	P xy & ¬ P yx	1 D_{PP}
3	PP yx	ass
4	P yx & ¬ P xy	3 D_{PP}
5	P xy & ¬ P xy	(2, 4 simp) conj
6	¬ PP yx	3-5 IP
7	PP xy ⇒ ¬ PP yx	1-6 CP

PP xy & PP yz ⇒ PP xz

1.	PP xy & PP yz	ass
2.	(P xy & ¬P yx) & (P yz & ¬P yz)	1 D_{PP}
3.	P xy & P yz	2 simp
4.	P xz	3, M3 MP
5.	P zx	ass
6.	P xz & P zx	4,5 conj
7.	x = z	6, M2 MP
8.	PP yz	1 simp
9.	PP yz ⇒ P yz	Th UI
10.	P yz	8,9 MP
11.	¬P yx	2 simp
12.	¬P yz	7,11 Id
13.	P yz & ¬P yz	10,12 conj
14.	¬ P zx	5-13 CP
15.	P xz & ¬ P zx	4, 14 conj
16.	PP xz	15 D_{PP}
17.	PP xy & PP yz ⇒ PP xz	1-17 CP

Equivalent axiomatization

- P. Simons in ‘Parts’
  - ¬PP xx
  - PP xy ⇒ ¬ PP yx
  - PP xy & PP yz ⇒ PP xz
  - P xy $=_{df}$ P xy & ¬ x = y

Summary

Usage of the word part in natural language

1. The handle is part of the cup.
2. This cap is part of my pen.
3. The left half is your part of the cake.
4. The US is part of North America.
5. The contents of this bag is only part of what I bought.
6. That corner is part of the living room.
7. The outermost points are part of the perimeter.
8. The first act was the best part of the play.

Formal axiomatic theories of mereology

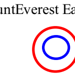
- Formulated in standard first or second order logic
- Include special symbols which are supposed to designate mereological relations or functions
- Some relations are treated as *primitives*: they are not defined. Axioms stipulating their logical properties are included the theory
- Other relations are *defined in terms of the primitives*

Formal axiomatic theories of mereology (2)

- A consistent formal theory has *models*:
  - models are collections of individuals
  - which satisfy the axioms of the theory when the primitives are interpreted in a certain way with respect to these individuals.

The non-logical primitive

- P xy
- Intended interpretation: x is a part of y
- examples
  - P your-hand you
  - P MountEverest Earth
  - P xy



Axioms of Ground mereology - **M**

- M1 P xx
- M2 P xy & P yx ⇒ x = y
- M3 P xy & P yz ⇒ P xz

Models

- An algebraic structure *A=(S,R)* is a model of a set of axioms with a single non-logical primitive *P* if and only if
  - The non-logical primitive *P* is interpreted as the the relation *R* of the structure
  - The axioms are true for each assignment of the variables with entities of the domain *S*

Ground mereology - **M**

- Axioms
  - M1 P xx
  - M2 P xy & P yx ⇒ x = y
  - M3 P xy & P yz ⇒ P xz
- Defined relations:
  - Overlap
  - Underlap
  - Proper part

Some Theorems of Ground Mereology

- M |-- O xx
- M |-- O xy ⇒ O yx
- M |-- P xy ⇔ (PP xy or x=y)
- M |-- ¬PP xx
- M |-- PP xy ⇒ ¬ PP yx
- M |-- PP xy & PP yz ⇒ PP xz
- M |-- P xy & ¬ (x=y) ⇒ PP xy

Assignments due by Sep. 10 Part 1

- Prove
  - M |-- (z)(P zx ⇔ P zy) ⇔ x = y
  - M |-- P xy ⇒ (z)(O zx ⇒ O zy)