ROUGH MEREOLOGY: A NEW PARADIGM FOR APPROXIMATE REASONING

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Abstract

We are concerned with formal models of reasoning under uncertainty. Many approaches to this problem are known in the literature e.g. Dempster-Shafer theory, bayesian-based reasoning, belief networks, fuzzy logics etc. We propose rough mereology as a foundation for approximate reasoning about complex objects. Our notion of a complex object includes approximate proofs understood as schemes constructed to support our assertions about the world on the basis of our incomplete or uncertain knowledge.

1 Introduction

We present a formal model of approximate reasoning about processes of synthesis of complex systems. First ideas of this approach have been presented in [15], [24], [25], [27], [28], [29], [30], [31]. Our research has been stimulated by the demand for solutions of the following groups of problems, estimated in [1] to be crucial for the progress in the area of automated design and manufacturing. These groups of problems are concerned with the treatment of:

Group 1. Poorly defined, poorly understood or incomplete design specifications.

Group 2. Negotiations among interacting goals and constraints.

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Group 3. Decomposition of problems into subproblems (including the problem of formation of a hierarchical scheme for solving the problem).

Group 4. Adaptation problems (including redesign and reuse problems).

Group 5. Problems of knowledge representation and reasoning about knowledge (including mapping of functions to structures and evaluating partial solutions at different levels of the synthesis scheme).

Design as well as manufacturing processes involve the space of specifications and the space of structures. These spaces are present in our approach at each local process site and they meet each other at the inventory level where primitive (indecomposable) specifications are converted into primitive (inventory) constructs.

Our analysis can be applied to the following fields concerned with complex systems:

Field 1. Computer-aided design [1], [32], [41] or computer-aided manufacturing [1], [4], [14], [41]. In this field, a complex system is synthesized , or designed, from elementary subsystems.

Field 2. Adaptive control of complex systems [13], [18], [28]. In this field, given specification (constraint) is maintained by adaptive adjustment of specifications for some subsystems.

Field 3. Business re-engineering [2], [22] (including software reuse). In this area, a complex system is adaptively modified according to a current requirement.

Field 4. Cooperative and distributive problem solving [7], [8], [9], [10], 14], [32], [41], [47-48]. In this field a complex system of local agents is organized from a set of agents in order to synthesize a solution to a problem.

The accessible knowledge on the basis of which constructs in the synthesis process are selected and classified (evaluated) is as a rule incomplete, poorly defined, or inconsistent. In consequence, we are bound to eveluate the basic ingredients of the synthesis process approximately only, in terms of values of some uncertainty measures which express a degree in which a given construct satisfies a given specification and in terms of some functors which propagate uncertainty measures along the synthesis scheme.

Many formal models of approximate reasoning are described in the literature e.g. Dempster-Schafer theory of evidence [33], [34], [37], bayesian reasoning [23], [34], belief networks [23], [34], many-valued logics [11], and fuzzy logics [11], non-monotonic logics [34] and neural network logics [17].

We can extract from these formal models a general scheme for approximate reasoning.

It is not suprising that this scheme encompasses classical models of reasoning adopted in mathematical logic [19].

The scheme for approximate reasoning can be represented by the following tuple

$$Appr\_Reas = (Ag, Link, U, St, Dec\_Sch, O, Inv, Unc\_mes, Unc\_prop)$$

where

(i) The symbol Ag denotes the set of agents (or agent names).

(ii) The symbol $Link$ denotes a finite set of non-empty strings over the alphabet Ag; for $v(ag) = ag_1 ag_2 ... ag_k ag \in Link$, we say that $v(ag)$ defines an *elementary synthesis scheme* $synt(ag_1, ag_2, .., ag_k, ag) = synt(v(ag))$ with the *root* ag and the *leaf agents* $ag_1, ag_2, ..., ag_k$. The intended meaning of $v(ag)$ is that the agents $ag_1, ag_2, .., ag_k$ are the children of the agent ag which can send to ag some constructs for assembling a complex artifact. The relation $ag \leq ag'$ iff ag is a leaf agent in $synt(v(ag))$ for some $v(ag)$ is usually assumed to be at least an ordering of Ag into a type of acyclic graph; we assume for simplicity that $(Ag, \leq)$ is a tree with the root $root(Ag)$ and leaf agents in the set $Leaf(Ag)$.

(iii) The symbol U denotes the set $\{U(ag) : ag \in Ag\}$ of universes of discourse (universes of constructs) of agents.

(iv) The symbol St denotes the set $\{St(ag) : ag \in Ag\}$ of standard sets of agents: for $ag \in Ag$, the set $St(ag) = \{st(ag)_i\} \subseteq U(ag)$ is the set od *standard constructs* (*objects*) of the agent ag.

(v) The symbol O denotes the set $\{O(ag) : ag \in Ag\}$ of *operations* with $O(ag) = \{o_i(ag)\}$ the set of *operations at ag*.

(vi) The symbol Dec_Sch denotes the set of *decomposition schemes*, a particular decomposition scheme dec_sch_j is a tuple

$$(\{st(ag)_j : ag \in Ag\}, (o_j(ag) : ag \in Ag\})$$

which satisfies the property that if $v(ag) = ag_1 ag_2 ... ag_k ag$ then

$$o_j(ag)(st(ag_1)_j, st(ag_2)_j, .., st(ag_k)_j) = st(ag)_j.$$

The intended meaning of dec_sch_j is that when any child ag_i of ag submits the standard construct $st(ag_i)_j$ then the agent ag assembles from

$$st(ag_1)_j, st(ag_2)_j, .., st(ag_k)_j$$

the standard construct $st(ag)_j$ by means of the operation $o_j(ag)$. The rule dec_sch_j establishes therefore a decomposition scheme of any standard construct at the agent $root\,(Ag)$ into a set of consecutively simpler standards at all other agents. The standard constructs of leaf agents are primitive standards. We can regard the set of decomposition schemes as a skeleton about which the approximate reasoning is organized. Any rule dec_sch_j conveys a certain knowledge that standard constructs are synthesized from specified simpler standard constructs by means of specified operations. This ideal knowledge is a reference point for real synthesis processes in which we deal as a rule with constructs which are not standard: in adaptive tasks, for instance, we process new, unseen yet, constructs (objects, signals).

(vii) The symbol Inv denotes the *inventory set* of primitive constructs.

(viii) The symbol Unc_mes denotes the set $\{Unc\_mes(ag) : ag \in Ag\}$ of *uncertainty measures of agents*, where $Unc\_mes(ag) = \{\mu_j(ag)\}$ and $\mu_j(ag) \subseteq U(ag) \times U(ag) \times V(ag)$ is a relation (possibly function) which determines a distance between constructs in $U(ag)$ valued in a set $V(ag)$; usually, $V(ag) = [0, 1]$, the unit interval.

(ix) The symbol Unc_prop denotes the set of *uncertainty propagation rules* $\{Unc\_prop(v(ag)) : v(ag) \in Link\}$; for $v(ag) = ag_1ag_2...ag_kag \in Link$, we have in $Unc\_prop(v(ag))$ the functions $f_j : V(ag_1) \times V(ag_2) \times ... \times V(ag_k) \longrightarrow V(ag)$ such that

if $\mu_j(ag_i)(x_i, st(ag_i)_j) = \varepsilon_i$ for $i = 1, 2, .., k$

$$\text{then } \mu_j(ag)(o_j(x_1, x_2, .., x_k), st(ag)_j) = \varepsilon \geq f_j(\varepsilon_1, \varepsilon_2, .., \varepsilon_k).$$

The functions f_j propagate uncertainty measures from children of ag to ag. The process of synthesis begins at leaf agents which receive primitive constructs and calculate their distances from their respective standards; then the primitive constructs are sent to the parent nodes of leaf agents along with vectors of distance values. The parent nodes synthesize complex constructs from the sent primitives and apply the uncertainty propagating functions in order to calculate from the sent vectors the new vectors of distances from their respective standards. Finally, the root agent $root(Ag)$ receives the constructs from its children from which it assembles the final construct and calculates the distances of this construct from the root standards. On the basis of the found values, the root agent classifies the final construct.

The above very general scheme is adapted to the particular cases. We would like to interpret this scheme taking as a particular instance the case of a fuzzy controller [11]. In its version due to Mamdani [18], in its simplest form, we have two agents: *input, output*, and standards of agents are expressed in terms of linguistic labels like*positively small, negative, zero* etc. Operations of the agent *output* express the control rules of the controller e.g. the symbol $o(positively\ small, negative) = zero$ is equivalent to the control rule of the form if $st(input)_i$ is *positively small* and $st(input)_j$ is *negative* then $st(output)_k$ is *zero*. Uncertainty measures of agents are introduced as fuzzy membership functions [11], [46] of the fuzzy sets corresponding to standards i.e. linguistic labels. An input construct (signal) $x(input)$ is fuzzified i.e. its distances from input standards are calculated and then the fuzzy logic rules are applied [11]. By means of these rules uncertainty propagating functions are defined which allow for calculating the distances of the output construct $x(output)$ from the output standards. On the basis of this distances the construct $x(output)$ is evaluated by the defuzzification procedure.

Our approach is anchored in rough set theory [20]. This theory assumes that constructs in the universe of discourse are perceived by means of the available information and in consequence these constructs are perceived as collections of constructs which bear the same information about them. The resulting granularity of knowledge is responsible for vagueness of knowledge. We are not able therefore to perceive individual constructs but their collections; we cannot in consequence discuss the membership relation but only containment relation. The counterpart of the notion of a fuzzy membership function would be the notion of a partial containment.

The formal treatment of partial containment is provided by the notion of a rough inclusion [24], [27], [29]. Rough inclusions are construed as most general functional objects conveying the intuitive meaning of the relation of being a part in a degree. In particular, the relation of being a part in the greatest possible degree is the relation of being a (possibly, improper) part in the sense of mereology of Stanislaw Leśniewski [16]. We can regard therefore a rough inclusion as a measure of departing from a decomposition scheme represented by the induced model of mereology of Leśniewski.

In mereology of Leśniewski the notions of a subset and of an element are equivalent and therefore we can interpret rough inclusions as global fuzzy membership functions on the universe of discourse which satisfy certain general requirements responsible for their regular mathematical properties.

We take rough inclusions of agents as measures of uncertainty in their respective universes. We would like to make the following two remarks.

Remark 1.1. Any non-leaf agent *ag* is able to establish a local decomposition scheme of complex constructs in its universe into some simpler parts by means of its rough inclusion $\mu(ag)$ and the relation *part* (of being a (proper) part) in the induced model of mereology of Leśniewski.

Remark 1.2. The mereological relation of being a part is not transitive globally over the whole synthesis scheme as distinct agents use distinct mereological languages.

The process of synthesis of a complex system by a scheme of agents consists in our approach of the two communication stages viz. the top - down communication/negotiation process and the bottom - up communication process. We outline the two stages here.

In the process of top - down communication, a requirement Φ received by the scheme from an external source is decomposed into approximate specifications of the form

$$(\Phi(ag), \varepsilon(ag))$$

for any agent ag of the scheme. The intended meaning of the approximate specification $(\Phi(ag), \varepsilon(ag))$ is that a construct $x \in U(ag)$ satisfies $(\Phi(ag), \varepsilon(ag))$ iff there exists a standard $st(ag)$ with the properties that $st(ag)$ satisfies the predicate $\Phi(ag)$ and

$$\mu(ag)(x, st(ag)) \geq \varepsilon(ag).$$

The uncertainty bounds of the form $\varepsilon(ag)$ are defined by the agents viz. the root agent *root(Ag)* chooses $\varepsilon(root(Ag))$ and $\Phi(root(Ag))$ as such that according to it any construct x satisfying $(\Phi(root(Ag), \varepsilon(root(Ag))$ should satisfy the external requirement Φ in an acceptable degree; the other agents choose their approximate specifications in negotiations within each elementary scheme $synt(v(ag))$ for $v(ag) \in Link$. The result of the negotiations is succesful when there exists a decomposition scheme dec_sch_j such that for any $v(ag) \in Link$, where $v(ag) = ag_1 ag_2 ... ag_k ag$, from the conditions $\mu(ag_i)(x_i, st(ag_i)_j) \geq \varepsilon(ag_i)$ and $st(ag_i)_j$ satisfies $\Phi(ag_i)$ for $i = 1, 2, .., k$, it follows that $\mu(ag)(o_j(x_1 x_2, .., x_k), st(ag)_j) \geq \varepsilon(ag)$ and st(ag)$_j$ satisfies $\Phi(ag)$.

The uncertainty bounds $\varepsilon(ag)$ are evaluated on the basis of uncertainty propagating functions whose approximations are extracted from information systems of agents.

Any leaf agent realizes its approximate specification by choosing in the subset $Inv \cap U(ag)$ of the inventory of primitive constructs a construct satisfying this specification.

The bottom-up communication consists of agents sending to their parents the chosen constructs and vectors of their rough mereological distances from the standards. The root agent $root(Ag)$ assembles the final construct.

Our approach is analytic in the sense that all objects necessary for the synthesis process are extracted from the empirical knowledge of agents represented in their information systems; it is also intensional in the sense that rules for propagating uncertainty are local as they depend on a particular elementary synthesis scheme and on a particular local standard.

Our presentation is divided into five sections. Preliminary notions of the rough set theory and mereology of Leśniewski are collected in Section 2 and Section 3. In Section 4 rough mereology is introduced in the form of the logic L_{rm}. Properties of models of L_{rm}, including properties of rough inclusions, are studied in Section 5. The final Section 6 brings a more detailed analysis of approximate reasoning by a system of distributed agents.

2 Preliminaries: Rough Set Theory

The formalization of vagueness within the framework of rough set theory is based on the assumption that objects are perceived by means of the information about them encompassed in a set of available features or attributes [20]; this informal idea leads to the notion of the information system.

An *information system* is a pair $\mathsf{A} = (U, A)$ where U is a finite set called the *universe of objects* and A is a finite set of *attributes*; any attribute $a \in A$ is a mapping on the universe U. We denote by the symbol V_a the range of the attribute a; the set V_a is called the *value set* of a. We let $V = \cup\{V_a : a \in A\}$.

In consequence of the above assumption some objects may become indiscernible. For an object $x \in U$ we define for a set $B \subseteq A$ the *information vector* $Inf_B(x) = \{(a, a(x)) : a \in B\}$. We say that objects $x, y \in U$ are *B-indiscernible* when $Inf_B(x) = Inf_B(y)$; the *B-indiscernibility relation* $IND(B)$ is defined as follows: $IND(B) = \{(x, y) \in U \times U : Inf_B(x) = Inf_B(y)\}$. The relation $IND(B)$ is an equivalence relation and we denote by the symbol $[x]_B$ the equivalence class of this relation which contains x. We will use the term *concept* for subsets of the universe U; for a concept $X \subseteq U$ we define the two approximations of X relative to a set $B \subseteq A$:

$$\underline{B}X = \{x \in U : [x]_B \subseteq X\} \text{ and } \overline{B}X = \{x \in U : [x]_B \cap X \neq \emptyset\}.$$

The concept $\underline{B}X$ is called the *B-lower approximation* of X and the concept $\overline{B}X$ is called the *B-upper approximation* of X. We collect in the following proposition (cf.[20]) the basic properties of approximations of X which follow directly from their definitions.

Proposition 1 *(i)* $\underline{B}X \subseteq X \subseteq \overline{B}X$*;*

(ii) $\underline{B}\ \underline{B}X = \underline{B}X$ *and* $\overline{B}\overline{B}X = \overline{B}X$*.*

□

The difference $BN_B(X) = \overline{B}X - \underline{B}X$ is called the *B-boundary region* of X. In the case when $BN_B(X) = \emptyset$ the concept X is said to be *B-exact*, otherwise X is *B-rough*. The concepts $\overline{B}X$, $\underline{B}X$ and $BN_B(X)$ have clear epistemic interpretation viz. the concept $\overline{B}X$ collects all objects which belong certainly in X, the concept $U - \overline{B}X$ collects all objects which certainly do not belong in X and the concept $BN_B(X)$ collects all objects which are vague with respect to X i.e. have representatives both in X and in the complement of X. It follows that $BN_B(X)$ is a non-sharp boundary of X in the sense of Frege (cf.[20]).

Given a concept X, the numerical characterization of a degree in which an object x belongs in the concept X relative to the knowledge represented by an attribute set $B \subseteq A$ is provided by the rough membership function $\mu_{X,B}$ [21]. For $B \subseteq A$, $X \subseteq U$ and $x \in U$, we let

$$\mu_{X,B}(x) = \frac{\|X \cap [x]_B\|}{\|[x]_B\|}$$

where $\|Z\|$ denotes the cardinality of a set Z. In the case when $B = A$ we use the symbol μ_X instead of the symbol $\mu_{X,A}$. The following proposition [21] collects the basic properties of the rough membership functions.

Proposition 2 *The rough membership functions of the form* $\mu_{X,B}$ *have the following properties*

(i) $\mu_{X,B}(x) = 1$ *iff* $x \in \underline{B}X$*;*

(ii) $\mu_{X,B}(x) = 0$ *iff* $x \in \overline{B}X$*;*

(iii) $0 < \mu_{X,B}(x) < 1$ *iff* $x \in \overline{B}X - \underline{B}X$;

(iv) if $(x, y) \in IND(B)$ *then* $\mu_{X,B}(x) = \mu_{X,B}(y)$*;*

(v) $\mu_{X,B}(x) = 1 - \mu_{U-X,B}(x)$*;*

(vi) $\mu_{X\cup Y,B}(x) \geq \max\{\mu_{X,B}(x), \mu_{Y,B}(x)\}$;

(vii) $\mu_{X\cap Y,B}(x) \leq \min\{\mu_{X,B}(x), \mu_{Y,B}(x)\}$;

(viii) for any pairwise disjoint collection P of concepts,

$$\mu_{\cup P,B}(x) = \sum\{\mu_{Y,B}(x) : Y \in P\}.$$

$\square$

We extend the notion of a rough membership function to a *standard rough inclusion* μ_U on the power set $\exp(U)$ of U. To this end, we define $\mu_U : \exp(U) \times \exp(U) \longrightarrow [0, 1]$ by letting

$$\mu_U(X, Y) = \frac{\|X \cap Y\|}{\|X\|} \text{ in case } X \neq \emptyset \text{ and } \mu_U(\emptyset, Y) = 1.$$

We denote by the symbol *Stand* the class of pairs of the form (U, μ_U) where U is a finite set and μ_U is the standard rough inclusion on the set U.

The reader will find in [26], [35], [36], [38], [39] a deep discussion of rough set-theoretic tools for decision rules generation and for synthesis of adaptive decision systems.

3 Preliminaries: Mereology of Leśniewski

The importance for logic of the fundamental study of relations of being a part was already stressed by Aristotle. The first modern mathematical system based on the notion of a relation of being a (proper) part was proposed by Stanislaw Leśniewski [16]. We recall here the basic notions of the mereological system of Leśniewski; in the next section the mereological system of Leśniewski will be extended to the system of approximate mereological calculus called *rough mereology.*

We consider a finite set U; we assume that U is non-empty. A binary relation π on the set U will be called the *relation of being a (proper) part* in the case when the following conditions are fulfilled

(P1) (irreflexivity) for any $x \in U$, it is not true that $x \pi x$;

(P2) (transitivity) for any triple $x, y, z \in U$, if $x \pi y$ and $y \pi z$, then $x \pi z$.

It follows obviously from (P1) and (P2) that the following property holds

(P3) for any pair $x, y \in U$, if $x \pi y$ then it is not true that $y \pi x$.

In the case when $x \pi y$ we say that the object x is a (*proper*) *part* of the object y. The notion of being (possibly) an improper part is rendered by the notion of an ingredient [16]; for objects $x, y \in U$, we say that the object x is a π-*ingredient* of the object y when either $x \pi y$ or $x = y$. We denote the relation of being a π-ingredient by the symbol $ingr(\pi)$; hence we can write

(I1) for $x, y \in U$, $x\ ingr(\pi)\ y$ iff $x \pi y$ or $x = y$.

It follows immediately from the definition that the relation of being an ingredient has the following properties:

(I2) (reflexivity) for any $x \in U$, we have $x\ ingr(\pi)\ x$;

(I3) (weak antisymmetry) for any pair $x, y \in U$, if $x\ ingr(\pi)\ y$ and $y\ ingr(\pi)\ x$ then $x = y$;

(I4) (transitivity) for any triple $x, y, z \in U$, if $x\ ingr(\pi)\ y$ and $y\ ingr(\pi)\ z$ then $x\ ingr(\pi)\ z$.

We will call any pair (U, π) where U is a finite set and π a binary relation on the set U which satisfies the conditions (P1) and (P2) a *pre-model of mereology.*

We now recall the notions of a set of objects and of a class of objects [16]. For a given pre-model (U, π) of mereology and a property m which can be attributed to objects in U, we will say that an object x is an object m (x *object* m, for short) when the object x has the property m. The property m will be said to be non-void when there exists an object $x \in U$ such that x *object* m. Consider a non-void property m of objects in a set U where (U, π) is a pre-model of mereology.

An object $x \in U$ is said to be a *set of objects with the property* m when the following condition is fulfilled:

(SETm) for any $y \in U$, if y *object* m and $y\ ingr(\pi)\ x$ then there exist $z, t \in U$ with the properties: $z\ ingr(\pi)\ y$, $z\ ingr(\pi)\ t$, $t\ ingr(\pi)\ x$ and t *object* m.

We will use the symbol x *set* m to denote the fact that an object x is a set of objects with the property m.

Assume that x *set* m; if, in addition, the object x satisfies the condition

(CLm) for any $y \in U$, if y *object* m then $y\ ingr(\pi)\ x$ then we say that the object x is *a class of objects with the property* m and we denote this fact by the symbol x *class* m. We will say that a pair (U, π) is a model of mereology when the pair (U, π) is a pre-model of mereology and the condition

(EUC) for any non-void property m of objects in the set U, there exists a unique object x such that x *class* m holds.

The following proposition [16] recapitulates the fundamental metamathematical properties of mereology of Leśniewski; observe that in mereology there is no hierarchy of objects contrary to the Cantorian naive set theory. We denote for an object $x \in U$ by the symbol $ingr(x)$ the property of being an ingredient of x (non-void in virtue of (I2)) and for a property m, we denote by the symbol $s(m)$ the property of being a set of objects with the property m.

Proposition 3 *For any $x \in U$,*

(i) x class $(ingr(x))$;

(ii) x class$(s(m))$ iff x class m;

(iii) x set $(s(m))$ iff x set m.

□

A more general proposition will be proved in Section 4. We finally recall the notions of an element and of a subset in mereology of Leśniewski. For $x, y \in U$, we will say that

(SUB) the object x is a *subset* of the object y when for any $z \in U$,

if z $ingr(\pi)$ x then z $ingr(\pi)$ y

and

(EL) the object x is an element of the object y when there exists a non-void property m such that x *object* m and y *class* m.

The following proposition which is a direct consequence of (I4) and Proposition 1(i) in Section 3 establishes the fact that in any model of mereology the notion of a subset is equivalent to the notion of an element.

Proposition 4 *Assume that a pair (U, π) is a model of mereology. Then the following statements are equivalent for any pair $x, y \in U$*

(i) x $ingr(\pi)$ y;

(ii) the object x is an element of the object y;

(iii) the object x is a subset of the object y.

□

The reader will consult [43] for ramifications of mereology of Leśniewski and

[5], [6], [45] for a development of the mereological calculus based on the predicate "*connected to*".

4 Rough Mereology

An approximate mereological calculus called rough mereology has been proposed as a formal treatment of the hierarchy of relations of being a part in a degree. We begin with an exposition of rough mereological calculus in the form of a logic L_{rm}.

4.1 Syntax of L_{rm}

It will be the standard syntax of the predicate calculus [19] in which we will have the following basic ingredients:

Variables: $x, x_1, x_2,..., y, y_1, y_2,..., z, z_1, z_2, ...$ of type *set_element* and $r, r_1, r_2,...,$ $s, s_1, s_2, ...$ of type *lattice_element*;

Constants: ω of type *lattice_element*;

Predicate symbols, function symbols: $\leq$ of type (*lattice_element, lattice_element*) and μ of type (*set_element, set_element, lattice_element)*;

Auxiliary symbols: propositional connectives: $\vee$, $\wedge$, $\Longrightarrow$, $\neg$, quantifier symbols: $\forall$, $\exists$ and commas, parentheses.

Formulae: atomic formulae are of the form $\mu(x, y, r)$, $s \leq r$ and formulae are built from atomic formulae as in the predicate calculus.

Axioms: the following are axioms of L_{rm}

(A1) $\forall x.\mu(x, x, \omega)$;

(A2) $\forall x.\forall y.\{\mu(x, y, \omega) \Longrightarrow \forall s.\forall r.\forall z.[\mu(z, x, s) \wedge \mu(z, y, r) \Longrightarrow (s \leq r)]\}$;

(A3) $\forall x.\forall y.\{\mu(x, y, \omega) \wedge \mu(y, x, \omega) \Longrightarrow$

$$\forall s.\forall r.\forall z.[\mu(x, z, s) \wedge \mu(y, z, r) \Longrightarrow (s \leq r)]\};$$

(A4) $\exists x.\forall y.\mu(x, y, \omega)$;

(A5) $\forall x.\forall y.\{[\forall z.[[\exists u.\neg(\mu(z,u,\omega)) \wedge \mu(z,x,\omega)] \Longrightarrow$

$$\exists t.(\exists w.(\neg\mu(t,w,\omega)) \wedge \mu(t,z,\omega) \wedge \mu(t,y,\omega)] \Longrightarrow \mu(x,y,\omega)\};$$

and the axiom schemata $(A6)_n$ for n = 2,3,.... where

$(A6)_n$

$\forall x_1.\forall x_2....\forall x_n.\exists y.(\alpha_n(x_1,x_2,...,x_n,y) \wedge$

$$\beta_n(x_1,x_2,...,x_n,y) \wedge \gamma_n(x_1,x_2,...,x_n,y))$$

where

$$\alpha_n(x_1,x_2,..,x_n,y) : \forall z.\{[\exists t.(\neg\mu(z,t,\omega)) \wedge \mu(z,y,\omega)] \Longrightarrow$$

$$\exists x_i.\exists w.[(\exists u.(\neg\mu(w,u,\omega))) \wedge \mu(w,z,\omega) \wedge \mu(w,x_i,\omega)]\};$$

$$\beta_n(x_1,x_2,..,x_n,y) : \mu(x_1,y,\omega) \wedge \mu(x_2,y,\omega) \wedge ... \wedge \mu(x_n,y,\omega);$$

$$\gamma_n(x_1,x_2,..,x_n,y) :$$

$$\forall z.\{[\alpha_n(x_1,x_2,..,x_n,z) \wedge \beta_n(x_1,x_2,..,x_n,z)] \Longrightarrow \mu(y,z,\omega)\}.$$

4.2 Semantics of L_{rm}

We will call an *interpretation* of L_{rm} a triple $M = (U^M, L^M, F^M)$ where U^M is a finite set, L^M is a (complete) lattice with the lattice partial order $\leq^M$and with the greatest element Ω^M and F^M is a mapping which assigns to constants and predicate symbols of L_{rm} their denotations in M in the following manner: $F^M(\omega) = \Omega^M$, $F^M(\leq) = \leq^M$and $F^M(\mu) = \mu^M \subseteq U^M \times U^M \times L^M$, where the relation $\mu^M \subseteq U^M \times U^M \times L^M$ is a function i.e. $\mu^M : U^M \times U^M \longrightarrow L^M$.

An M-value assignment g is a mapping which assigns to any variable x of L_{rm} of type *set_element* the element $g(x) \in U^M$ and to any variable r of L_{rm} of type *lattice_element* the element $g(r) \in L^M$. For an M-value assignment g, a variable x of L_{rm} of type *set_element* and an element $u \in U^M$, we denote by the symbol $g[u/x]$ the M-value assignment defined by the conditions: $g[u/x](v) = g(v)$ in case $v \neq x$ and $g[u/x](x) = u$; the same convention will define $g[p/r]$ in case of a variable r of type *lattice_element* and $p \in L^M$.

For a formula α of L_{rm}, we denote by the symbol $[\alpha]^{M,g}$ the meaning of the formula α in the model M relative to an M-value assignment g by the following conditions

(M1) $[\mu(x,y,r)]^{M,g} = true$ iff $\mu^M(g(x), g(y)) = p$ for some $p \geq^M g(r)$;

(M2) $[s \leq r]^{M,g} = true$ iff $g(s) \leq^M g(r)$;

(M3) $[\alpha \vee \beta]^{M,g} = true$ iff $[\alpha]^{M,g} = true$ or $[\beta]^{M,g} = true$;

(M4) $[\neg\alpha]^{M,g} = true$ iff $[\alpha]^{M,g} = false$;

(M5) $[\exists x.\alpha]^{M,g} = true$ iff there exists $u \in U^M$ such that $[\alpha]^{M,g[u/x]} = true$;

(M6) $[\exists r.\alpha]^{M,g} = true$ iff there exists $p \in L^M$ such that $[\alpha]^{M,g[p/r]} = true$.

It follows that the intended meaning of a formula $\mu(x,y,r)$ is that "the object x is a part of the object y in degree at least r".

A formula α is *true in an interpretation* M iff α is M,g-true (i.e. $[\alpha]^{M,g} = true$) for any M-value assignment g.

An interpretation M is a *model* of L_{rm} iff all axioms (A1)-(A6) are true in M.

4.3 Deduction rules

We will give the basic deduction rules for L_{rm}; recall that a deduction rule in the form $\frac{\alpha,\beta,...}{\psi}$ is said to be *valid in a model* M iff for any M-value assignment g if the premises $\alpha,\beta,...$ are M,g - true then the conclusion ψ is M,g-true. The deduction rule is *valid* when it is valid in any model M of L_{rm}. We have the following deduction rules

(D1) $\frac{\mu(x,y,\omega),\mu(y,z,\omega)}{\mu(x,z,\omega)}$

(D2) $\frac{\mu(y,z,\omega),\neg\mu(y,x,\omega)}{\neg\mu(z,x,\omega)}$;

(D3) $\frac{\mu(x,y,\omega),\neg\mu(z,y,\omega)}{\neg\mu(z,x,\omega)}$.

Proposition 5 *Deduction rules (D1)-(D3) are valid.*

Proof

We consider a model $M = (U^M, L^M, F^M)$ along with an M-value assignment g. We prove the validity of (D1); proofs for (D2), (D3) go along similar lines. Assume that $[\mu(x,y,\omega)]^{M,g} = true = [\mu(y,z,\omega)]^{M,g}$ i.e. $\mu^M(g(x), g(y)) =$

$\Omega^M = \mu^M(g(y), g(z))$. By the truth of (A2) in M, we have $\mu^M(g(x), g(z)) \geq^M \mu^M(g(x), g(y))$ hence $\mu^M(g(x), g(z)) = \Omega^M$ i.e. $[\mu(x, z, \omega)]^{M,g} = true$. This concludes the proof.

□

4.4 The consistency of axioms

We show the consistency of the axiom system (A1)-(A6) by revealing a class of models of L_{rm}. We denote by *Stand* the class consisting of pairs (U, μ_U) where U is a finite set and μ_U is the standard rough inclusion on the set $\exp(U)$. For a pair $M = (U, \mu_U)$, we let $L^M = [0, 1]$, the unit interval , $\leq^M$=the natural linear ordering on [0,1] , $\mu^M = \mu_U$ and $U^M = \exp(U)$.Then we denote by *Stand_Mod* the class of triples $M^* =(U^M, L^M, F^M)$ where $M = (U, \mu_U)$ and $F^M(\omega) = 1$, $F^M(\leq) =\leq^M$ and $F^M(\mu) = \mu_U$. We have the following

Proposition 6 *Any $M^* =(U^M, L^M, F^M)$ in Stand_Mod is a model of L_{rm}.*

Proof

We consider $M^* =(U^M, L^M, F^M)$ in *Stand_Mod* and an M^*-valued assignment g. Concerning (A1), we have $\mu_U(g(x), g(x)) = 1$. Concerning (A2), when $\mu_U(g(x), g(y)) = 1$ then either $g(x) = \emptyset$ and $\mu_U(g(z), g(y)) \geq 0 = \mu_U(g(z), g(x))$ in case $g(z) \neq \emptyset$ or $\mu_U(g(z), g(y)) = 1 \geq \mu_U(g(z), g(x))$ in case $g(z) = \emptyset$. The case of (A3) is similar to that of (A2). The truth of (A4) is witnessed by the M^*-value assignment $g[\emptyset/x]$. For (A5), consider $g(x) = X \neq \emptyset, g(y) = Y$ (when $X = \emptyset$ there is nothing to prove) with the property that for any $Z \neq \emptyset$ with the property $\mu_U(Z, X) = 1$ there exists $T \neq \emptyset$ such that $\mu_U(T, Z) = 1 = \mu_U(T, Y)$. This implies that $\mu_U(X, Y) = 1$, otherwise we would obtain contradiction with the assumption by taking the M^*-valued assignment $g[X - Y/z]$. Finally, in case of (A6) for an integer n, given an M^*-valued assignment $g = g[X_1/x_1, X_2/x_2, .., X_n/x_n]$ one checks easily that $g[Y/y]$ where $Y = X_1 \cup X_2 \cup ... \cup X_n$ witnesses the truth of $(A6)_n$.

□

5 Rough inclusions

In this section we are concerned with the structure of models of L_{rm} induced by rough inclusions. We show that in any model of L_{rm} we have a canonical model of mereology of Leśniewski introduced by means of the rough inclusion of this model. We apply the Tarski idea of fusion of sets [43] in order to define in a model of L_{rm} the structure of a (complete) Boolean algebra which contains isomorphically the quasi-boolean structure (without the least element) corresponding to the model of mereology of Leśniewski. We show that the rough inclusion satisfies with respect to boolean operations of join and meet the same formal conditions which the rough membership function satisfies with respect to the set-theoretic operations of union and intersection.

We study relations of rough inclusions with many-valued logic and fuzzy logic; in particular, we show that when the rough inclusion is regarded as a fuzzy membership function then any fuzzy containment induced by a residual implication [11] is again a rough inclusion and moreover, the hierarchy of objects set by the induced model of mereology of Leśniewski is invariant under these fuzzy containment operators.

We are concerned also with the problem of consistency of deduction rules of the form

$$(\mathrm{D}_f)\ \frac{\mu(x,y,r),\mu(y,z,s)}{\mu(x,z,f(r,s))}$$

where f is a functional symbol of type (*lattice_element*, *lattice_element*, *lattice_element*).

We demonstrate the consistency of (A1)-(A6)+(D_f) by revealing a class of models in which the deduction rule (D_f) is valid under an appropriate interpretation of f.

5.1 Rough inclusions: reduced models and Leśniewski's mereology

Given a model M of L_{rm}, $M = (U^M, L^M, F^M)$, we will call the function $\mu^M : U^M \times U^M \longrightarrow L^M$ the M-rough inclusion. We define a relation $congr(\mu^M)$ on the set U^M by letting for $u, w \in U^M$: $u\ congr(\mu^M)\ w$ iff $\mu^M(u,w) = \Omega^M$ $= \mu^M(w,u)$. The following proposition, whose proof follows immediately by (A2) and (A3) and is therefore omitted, establishes the basic properties of the relation $congr(\mu^M)$ and demonstrates it to be a μ^M-congruence.

Proposition 7 *The relation $congr(\mu^M)$ is an equivalence relation on the set U^M and we have*

(i) if $u\ congr(\mu^M)\ w$ then $\mu^M(v,w) = \mu^M(v,u)$;

(ii) if u congr(μ^M) w then $\mu^M(u,v) = \mu^M(w,v)$

for any triple $u, v, w \in U^M$.

□

For $u \in U^M$ we denote by u_μ the equivalence class of the relation $congr(\mu^M)$ which contains u. It follows from prop 1 in Section 5 that the rough inclusion can be factored throughout the relation $congr(\mu^M)$ i.e. we define the quotient set $U^M_\mu = U^M/congr(\mu^M)$ and the quotient function

$$\mu^M_\sim : U^M_\mu \times U^M_\mu \longrightarrow L^M$$

by letting $\mu^M_\mu(u_\mu, w_\mu) = \mu^M(u,w)$; clearly, the pair $(U^M_\mu, \mu^M_\sim)$ introduces a model $M_\sim$ of L_{rm}. In the sequel we will always work with a fixed reduced model $M_\sim$. We denote by the symbol n_μ the *null object* i.e. the object existing in virtue of (A4) and such that $\mu^M_\sim(n_\mu, w_\mu) = \Omega^M$ for any $w_\mu \in U^M_\mu$. We will write $u_\mu \neq_\mu n_\mu$ to denote the fact that the object u_μ is not the null object. Let us recall that the existence of a null object in a model of mereology of Leśniewski reduces the model to a singleton, as observed in Tarski [42]. In the sequel, for simplicity of notation, we will write μ in place of $\mu^M_\sim$, U in place of U^M_μ, u in place of u_μ etc. We will call the rough inclusion μ a *strict rough inclusion* when it satisfies the condition $\mu(x,n) = 0$ for any non-null object x; we observe that any standard rough inclusion is strict.

We now show how the rough inclusion μ introduces in U a model of mereology of Leśniewski. To this end, we define a binary relation $part(\mu)$ on the set U by letting

u $part(\mu)$ w iff $\mu(u,w) = \Omega^M$ and it is not true that $\mu(w,u) = \Omega^M$.

Then we have the following proposition whose straightforward proof is omitted

Proposition 8 *(i) the relation $part(\mu)$ satisfies the conditions (P1) and (P2);*

(ii) the relation ingr($part(\mu)$) satisfies the following for any pair $u, w \in U$:

$$u\ ingr(part(\mu))\ w \text{ iff } \mu(u,w) = \Omega^M.$$

□

It follows from the proposition above that $(U, part(\mu))$ is a pre-model of mereology. We now define in the model $M_\sim$ for any collection Ψ of objects in U, the notions of a set of objects in Ψ and of a class of objects in Ψ. We will say then that $u \in U$ is a *set of objects in Ψ*, *u set Ψ* for short, when

(S1) for any $w \neq_\mu n$ such that $w\ ingr(part(\mu)\ )\ u$ there exist $v \neq_\mu n$ and $t \in \Psi$ such that $v\ ingr(part(\mu)\ )\ w$, $v\ ingr(part(\mu))\ t$, $t\ ingr(part(\mu)\ )\ u$;

if in addition, we have

(S2) $t\ ingr(part(\mu)\ )\ u$ for any $t \in \Psi$;

(S3) for any t, if t satisfies (S1) and (S2) with Ψ then $u\ ingr(part(\mu)\ )\ t$

then we say that u is a *class of objects in* Ψ, $u\ class\ \Psi$ for short. It follows from (A6) that for any collection Ψ there exists a unique object u such that $u\ class\ \Psi$ and there exists objects of the form $set\ \Psi$. We have therefore

Proposition 9 *The pair* $(\ U - \{n\}, part(\mu) \uparrow (U - \{n\}) \times (U - \{n\})$ *is a model of mereology.*

□

We now show that in the model $M_\sim$ (fixed arbitrarily) of L_{rm} the following generalization of prop 1 from Section 3 holds. The symbol $Ingr(u)$ denotes the collection of $part(\mu)$-ingredients of u, and $Set\Psi$ stands for the property of being a set of objects in Ψ.

Proposition 10 *For any* $u \in U$ *we have*

(i) $u\ class\ Ingr(u)$;

(ii) if $u\ class\ Set\Psi$ *then* $u\ class\ \Psi$;

(iii) if $u\ set\ Set\Psi$ *then* $u\ set\ \Psi$;

(iv) if $u\ class\ \Psi$ *then* $u\ class\ Set\Psi$.

Proof

As proofs of (ii)-(iv) are carried out on similar lines , we observe that (i) follows immediately from the definition and we prove (ii) . To prove (ii), we assume that $u\ class\ Set\Psi$. Let $v \neq_\mu n$ and $v\ ingr(part(\mu)\ )\ u$. Clearly, $u\ set\ Set\Psi$ and thus in virtue of (S1) there exist $w, t \in U$ such that $w \neq_\mu n$, $w\ ingr(part(\mu)\ )\ v$, $w\ ingr(part(\mu)\ )\ t$, $t\ ingr(part(\mu)\ )\ u$ and $t\ set\ \Psi$. Hence, again by the truth of (S1), there exist $p, q \in U$ such that

$$p \neq_\mu n,\ p\ ingr(part(\mu))\ w,\ p\ ingr(part(\mu))\ q,\ q\ ingr(part(\mu))\ t \text{ and } q \in \Psi.$$

Then we have $q\ ingr(part(\mu)\ )\ u$ and thus $u\ set\ \Psi$. It follows that u satisfies (S1) with Ψ.

Now, we consider any $q \in \Psi$; clearly, $q\ set\ \Psi$ and by the truth of (S2) we have $q\ ingr(part(\mu)\ )\ u$ i.e. u satisfies (S2) with Ψ. Finally, assume that $w \in U$ satisfies the condition $\mu(u, w) < \Omega^M$; it follows by (S3) that either w does not satisfy (S1) or w does not satisfy (S2) with $Set\ \Psi$. We consider the two cases.

Case 1. w does not satisfy (S2). There exists $z \in U$, such that $\mu(z, w) < \Omega^M$ and $z\ set\ \Psi$.

Subcase 1a. We assume that for any $t \in U$ if $t \neq_\mu n$, $t\ ingr(part(\mu))\ z$ and $t \in \Psi$, then $t\ ingr(part(\mu))\ w$. We consider $t \in U$ such that $t \neq_\mu n$, and $t\ ingr(part(\mu))\ z$. As we have $z\ set\ \Psi$, there exist by (S1) $p, q \in U$ such that $p \neq_\mu n$, $p\ ingr(part(\mu))\ t$, $p\ ingr(part(\mu))\ q$, $q\ ingr(part(\mu))\ z$ and $q \in \Psi$. By our assumption, we have $q\ ingr(part(\mu))\ w$. Hence $p\ ingr(part(\mu))\ w$ and it follows from (A4) that $z\ ingr(part(\mu))\ w$, a contradiction.

It follows that we are left with

Subcase 1b.

There exists $v \in U$ such that $v \neq_\mu n$, $v\ ingr(part(\mu))\ z$, $v \in \Psi$ and $\mu(v, w) < \Omega^M$. But this means that w does not satisfy (S2) with Ψ.

Case 2. w does not satisfy (S1). Then clearly, it is not true that $w\ set\ \Psi$ i.e. w does not satisfy (S1) with Ψ.

It follows from Cases 1 and 2 that $u\ class\ \Psi$. This concludes the proof of (ii) and the proof of the proposition.

□

We now outline the boolean structure induced in the model $M_\sim$ by the rough inclusion μ. We first define, extending the idea of [16], the relation $ext\ (\mu)$; to this end, we let for $u, w \in U$:

$u\ ext\ (\mu)\ w$ iff it is not true that there exists $z \in U$ such that $z \neq_\mu n$, $z\ ingr(part(\mu))\ u$, and $z\ ingr(part(\mu))\ w$. Following the idea of Tarski [43], we define boolean operations $\vee^M$, $\wedge^M$, $\neg^M$ by letting for $u, w \in U$: $u \vee^M w$ is the class of objects in $\Psi(u, w)$ where $\Psi(u, w)$ is the collection of objects which contains an object t iff either $t\ ingr(part(\mu))\ u$ or $t\ ingr(part(\mu))\ w$; $u \wedge^M w$ is the class of objects in $\Phi(u, w)$ where $\Phi(u, w)$ is the collection of objects which contains an object t iff $t\ ingr(part(\mu))\ u$ and $t\ ingr(part(\mu))\ w$; finally, $\neg^M u$ is the class of objects in $\Xi(u)$ where $\Xi(u)$ is the collection of objects which contains an object t iff $t\ ext\ (\mu)\ u$.

We have the following proposition; the straightforward proof much in the spirit of the proof of Proposition 4 in Section 5 is omitted.

Proposition 11 *The operations $\vee^M$, $\wedge^M$, $\neg^M$ introduce into the set U the structure of a Boolean algebra viz. the following properties hold*

(i) $u\ congr(\mu)\ (\neg^M \neg^M\ u)$;

(ii) $(u \vee^M w)\ congr(\mu)\ (w \vee^M\ u)$;

(iii) $((u \vee^M w) \vee^M z)\ congr(\mu)\ (u \vee^M\ (w \vee^M\ z))$;

(iv) $(u \vee^M\ u)\ congr(\mu)\ u$;

(v) $(u \vee^M\ n)\ congr(\mu)\ u$;

(vi) $(u \vee^M\ (\neg^M u))\ congr(\mu)\ U$;

(vii) $(u \wedge^M w)\ congr(\mu)\ (w \wedge^M u)$;

(viii) $(u \wedge^M\ (w \wedge^M z))\ congr(\mu)\ ((u \wedge^M w) \wedge^M z)$;

(ix) $(u \wedge^M\ u)\ congr(\mu)\ u$;

(x) $(u \wedge^M\ (\neg^M u))\ congr(\mu)\ n$;

(xi) $(u \wedge^M U)\ congr(\mu)\ u$;

(xii) $\neg^M(u \vee^M w)\ congr(\mu)\ ((\neg^M u) \wedge^M (\neg^M w))$;

(xiii) $\neg^M(u \wedge^M w)\ congr(\mu)\ ((\neg^M u) \vee^M (\neg^M w))$;

(xiv) $(u \wedge^M (v \vee^M\ w))\ congr(\mu)\ ((u \wedge^M v) \vee^M (u \wedge^M w))$;

(xv) $(u \vee^M\ (v \wedge^M\ w))\ congr(\mu)\ ((u \vee^M v) \wedge^M (u \vee^M w))$;

(xvi) $(u \wedge^M (u \vee^M w))\ congr(\mu)\ u$;

(xvii) $(u \vee^M (u \wedge^M w))\ congr(\mu)\ u$.

□

We now observe that the rough inclusion μ behaves with respect to the boolean operations $\vee^M$ and $\wedge^M$ in the same way as the rough membership function behaves to the set-theoretic operations of the union and the intersection; the following proposition is therefore a far-reaching generalization of prop 2 (vi),

(vii) in Section 2 and it demonstrates that the operators max and min are the limiting operators in , respectively, the additive and the multiplicative cases, in the widest sense.

Proposition 12 *(i)* $\mu(u, v \vee^M w) \geq \max\ \{\mu(u,w),\ \mu(v,w)\}$;

(ii) $\mu(u \wedge^M v, t) \leq min\ \{\mu(u,t),\ \mu(v,t)\}$.

Proof It suffices to observe that $(u \wedge^M w)\ ingr(part(\mu))\ u$, $(u \wedge^M w)\ ingr\ (part(\mu))\ w$, $u\ ingr(part(\mu))\ (u \vee^M w)$ and $w\ ingr(part(\mu))\ (u \vee^M w)$ and to apply (A2).

□

5.2 Rough inclusion in the context of many-valued logic

In this section we will reveal some of the basic connections between rough mereology and many-valued logic [11], announced above. We recall that a *t-norm* $\top$ is a mapping $\top : [0,1] \times [0,1] \longrightarrow [0,1]$ which satisfies the conditions $\top(r,1) = r$, $\top(r,s) = \top(s,r)$, if $r \leq s$ then $\top(r,t) \leq \top(s,t)$ and $\top(r, \top(s,t)) = \top(\top(r,s),t)$. A *residual implication* $\overrightarrow{\top}$ induced by a t-norm $\top$ is a mapping $\overrightarrow{\top} : [0,1] \times [0,1] \longrightarrow [0,1]$ which satisfies the condition

$$\top(r,s) \leq t \text{ iff } r \leq \overrightarrow{\top}(s,t).$$

Clearly, when a t-norm $\top$ is a continuous mapping then we have a unique residual implication

$$\overrightarrow{\top}(s,t) = \sup\{r : \top(r,s) \leq t).$$

We consider a model $M_\sim$ of L_{rm}. As the induced model of mereology has the property that the notions of a set and of a subset are equivalent, we can interpret the value $\mu(u,w)$ as the value of a fuzzy membership function $\mu_w(u)$ in the sense of fuzzy set theory [46]. The partial containment is expressed in this theory [11] by means of a many-valued implication viz. for a given many-valued implication $I : [0,1] \times [0,1] \longrightarrow [0,1]$, the induced partial containment function $\sigma_I(u,w)$ is defined by the formula: $\sigma_I(u,w) = \inf\{I(\mu_u(z), \mu_w(z)) : z \in U\}$. We show that when the implication I is a residual implication $\overrightarrow{\top}$ induced by a continuous t-norm $\top$ then the resulting function $\sigma_\top$ is a rough inclusion and, moreover, the function $\sigma_\top$ preserves the relation $ingr(part(\mu))$ hence it induces the identity isomorphism of the corresponding boolean algebras. Our next proposition reads as follows.

Proposition 13 *For a continuous t-norm* $\top$ *and a model* $M_\sim$ *of* L_{rm} *with the strict rough inclusion* μ*, the function*

$$(i)\ \sigma_\top(u,w) = \inf\{\overrightarrow{\top}(\mu_u(z),\mu_w(z)) : z \in U\}$$

is a rough inclusion; moreover, we have

$$(ii)\ \sigma_\top(u,w) = 1 \text{ iff } \mu(u,w) = 1.$$

Proof We first observe that $\sigma_\top(u,u) = \inf\{\overrightarrow{\top}(\mu_u(z),\mu_u(z)) : z \in U\} = \inf\{1 : z \in U\} = 1$. Next, we assume that $\sigma_\top(u,w) = 1$; it follows that $\overrightarrow{\top}(\mu_u(z),\mu_w(z)) = 1$ for any $z \in U$ hence we have $\mu_u(z) = \top(1,\mu_u(z)) \leq \mu_w(z)$ and thus $\overrightarrow{\top}(\mu_v(t),\mu_w(t)) \geq \overrightarrow{\top}(\mu_v(t),\mu_u(t))$ for all v,t which by taking the infimum over all t yields $\sigma_\top(v,w) \geq \sigma_\top(v,u)$. We now assume that $\sigma_\top(u,w) = 1 = \sigma_\top(w,u)$. Hence we have $\overrightarrow{\top}(\mu_u(t),\mu_w(t)) = 1$ and $\overrightarrow{\top}(\mu_w(t),\mu_u(t)) = 1$ for any $t \in U$. It follows that $\mu_w(t) \geq \top(1,\mu_u(t)) = \mu_u(t)$ and, similarly, $\mu_u(t) \geq \mu_w(t)$ for any $t \in U$. We have therefore $\mu_u(t) = \mu_w(t)$ for any $t \in U$. For a given $v \in U$, we therefore obtain

$\sigma_\top(u,v) = \inf\{\overline{\top}(\mu_u(t),\mu_v(t)); t \in U\} = \inf\{\overrightarrow{\top}(\mu_w(t),\mu_v(t)) : t \in U\} = \sigma_\top(w,v)$.

We now consider $\sigma_\top(n,u)$ for $u \in U$; we have

$$\sigma_\top(n,u) = \inf\{\overrightarrow{\top}(\mu_n(z),\mu_u(z)) : z \in U\} = 1$$

(either z is the null-object and then $\overrightarrow{\top}(\mu_n(z),\mu_u(z)) = \overrightarrow{\top}(1,1) = 1$ or $z \neq_\mu n$ and $\overrightarrow{\top}(\mu_n(z),\mu_u(z)) = \overrightarrow{\top}(0,\mu_u(z)) = 1$). We have proved that $(U,\sigma_\top)$ is a model for (A1)-(A4).

We now digress from the proof that $\sigma_\top$ is a rough inclusion and we prove that (ii) holds. We first assume that $\sigma_\top(u,w) = 1$. Then we have $\overrightarrow{\top}(\mu_u(z),\mu_w(z))$ for any $z \in U$ hence $\overline{\top}(\mu_u(u),\mu_w(u)) = 1$ and this implies that $1 \leq \mu_w(u)$ i.e. $\mu(u,w) = 1$. We now assume that $\mu(u,w) = 1$; it follows by (A2) that $\mu(v,w) \geq \mu(v,u)$ for any $v \in U$ hence $\overrightarrow{\top}(\mu_u(v),\mu_w(v)) = 1$ for any $v \in U$ and it follows that $\sigma_\top(u,w) = 1$.

It follows that if we have a condition employing the formula $\sigma_\top(u, w) = 1$ and we replace in this condition the formula $\sigma_\top(u, w) = 1$ by the formula $\mu(u, w) = 1$ then we obtain the equivalent condition. From this remark it infer immediately that $\sigma_\top$ satisfies (A5) and (A6). This concludes the proof.

□

We denote by $M_\top$ the model of L_{rm} obtained from a model $M_\sim$ with a strict rough inclusion μ by replacing μ with $\sigma_\top$. By the symbol

$$Stand\_Mod(\top)$$

we denote the class of models of the form $M_\top$ where M is a standard model of L_{rm}.

We now prove the consistency of the deduction rule of the form (D_f); the symbol CON((A1)-(A6)+(D_f)) denotes the consistency of (D_f) i.e. the existence of the model of L_{rm} in which (D_f) is valid under a plausible interpretation of the function symbol f. We extend the syntax of the L_{rm} by adding a functional constant symbol f of type

$$(lattice\_element,\ lattice\_element,\ lattice\_element).$$

we extend accordingly the domain of F^M. Then we have

Proposition 14 *CON((A1)-(A6) +(D_f)); more specifically, the deduction rule (D_f) is valid in any model M in Stand_Mod($\top$) where $F^M(f) = \top$.*

Proof We assume that $\sigma_\top(u, v) = r$, $\sigma_\top(v, w) = s$. We have $r = \inf\{\overrightarrow{\top}(\mu_u(t), \mu_v(t)) : t \in U\}$ and $s = \inf\{\overrightarrow{\top}(\mu_v(t), \mu_w(t)) : t \in U\}$. Clearly, when $r = 1 = s$, the rule ($D_\top$) is the rule (D1). We consider some cases.

1. In the case when $r < 1$ and $s = 1$, we have $\mu_v(t) \leq \mu_w(t)$ for any $t \in U$ and thus

$$\sigma_\top(u, w) = \inf\{\overrightarrow{\top}(\mu_u(t), \mu_w(t)) : t \in U\} \geq$$

$$\inf\{\overrightarrow{\top}(\mu_u(t), \mu_v(t)) : t \in U\} = r = \top(r, 1).$$

2. In the case when $r = 1$ and $s < 1$ it is enough to consider a fixed $t \in U$ such that $r_t = \overrightarrow{\top}(\mu_u(t), \mu_v(t)) = 1$ and $s_t = \overrightarrow{\top}(\mu_v(t), \mu_w(t)) < 1$. We have $\mu_u(t) \leq \mu_v(t)$ and $\top(s_t, \mu_v(t)) = \mu_w(t)$ hence $\top(s_t, \mu_u(t)) \leq \mu_w(t)$ which implies that $\overrightarrow{\top}(\mu_u(t), \mu_w(t)) \geq s_t = \top(1, s_t) = \top(r_t, s_t)$; passing to infima over

t on both sides gives $\sigma_{\top}(u,w) \geq \inf\{\top(r_t,s_t) : t \in U\} \geq \top(\inf\{r_t : t \in U\}, \inf\{s_t : t \in U\}\} = \top(r,s)$.

3. It remains to consider the case when $r, s < 1$. We proceed as in the case 2 i.e. we take a fixed $t \in U$ and we consider r_t and s_t defined above. We show that we always have the inequality $\overrightarrow{\top}(\mu_u(t), \mu_w(t)) \geq \top(r_t, s_t)$ from which the inequality $\sigma_{\top}(u,w) \geq \top(r,s)$ follows by passing to infima on both sides. The case $r_t = 1 and s_t < 1$ has already been considered in case 2 and the case $r_t = 1 = s_t$ follows obviously. In the case $r_t < 1, s_t = 1$ we proceed in the same way as in the case 1. It remains to consider the case $r_t < 1, s_t < 1$. We have $\top(r_t, \mu_u(t)) = \mu_v(t)$ and $\top(s_t, \mu_v(t)) = \mu_w(t)$. It follows that $\top(s_t, \top(r_t, \mu_u(t))) = \mu_w(t)$ i.e. $\top(\top(r_t, s_t), \mu_u(t)) = \mu_w(t)$. It follows that $\overrightarrow{\top}(\mu_u(t), \mu_w(t)) \geq \top(r_t, s_t)$ and by taking the infimum over all t we obtain the inequality $\sigma_{\top}(u,w) \geq \inf\{\top(r_t,s_t) : t \in U\} \geq \top(\inf\{r_t : t \in U\}, \inf\{\mathrm{s}_t : t \in U\}) = \top(r,s)$.

This concludes the proof.

□

6 An Application: A Rough Mereology Based Distributed System For Synthesis of Approximative Solutions

We present a general scheme for synthesis of approximate solutions to a given requirement. We begin with introductory remarks which provide a motivation and explain our methodological assumptions.

6.1 Methodology

We begin with an example of a synthesis of a solution in a classical context. Consider a formula α: $p \longrightarrow p$ in the propositional calculus [19]. To give a formal proof of α requires a derivation of α from a system of axioms of the propositional calculus by means of allowed inference rules. An exemplary derivation of α is the following one represented in a sequence of steps.

Step 1. The instances (I1), (I2), (I3) of axiom schemata are taken:

(I1) $p \longrightarrow (p \longrightarrow p)$;

(I2) $(p \longrightarrow ((p \longrightarrow p) \longrightarrow p)) \longrightarrow ((p \longrightarrow (p \longrightarrow p)) \longrightarrow (p \longrightarrow p))$;

(I3) $p \longrightarrow ((p \longrightarrow p) \longrightarrow p)$.

Step 2. From (I2) and (I3) by applying the modus ponens (MP) inference rule the formula

(M) $(p \longrightarrow (p \longrightarrow p)) \longrightarrow (p \longrightarrow p)$

is obtained.

Step 3. From the formulas (M) and (I1) the formula (R) is obtained by applying the (MP) inference rule

(R) $p \longrightarrow p$.

The formula (R) is α and the derivation is concluded. The above derivation of the formula α can be regarded as a scheme for synthesis of a solution (a derivation) to the requirement α. In this scheme we can distinguish some specialized agents: *R, M, I1, I2, I3*. The agents perform specialized tasks and are involved in communication and negotiation processes which can be described in a sequence of stages.

Stage 1. The agent R (the *root agent of the scheme*) receives the formula α and decomposes it into some formulas β, γ (possibly non-uniquely) from which it can produce α by means of its operation (MP).

Stage 2. The agent R and agents *M, I1* negotiate the particular decomposition of α; in our example, the decomposition is chosen into (M) and (I1).

Stage 3. The agent *I1* is an *inventory* (*leaf*) agent : it is able to find a required formula in the inventory of instances of axiom schemata. The agent M can repeat the stage 1 with the formula (M) by negotiating with agents *I2* and *I3* the decomposition of (M) into formulas δ, ϵ from which M is able to produce (M) by means of its inference rule (MP). In our example, this decomposition is (I2), (I3).

Stage 4. The agents *I1, I2, I3* send the required negotiated formulas from the inventory to their parents. The agent M synthesizes the formula (M) and sends it to the parent R. The agent R produces the formula (R) and sends it as the solution satisfying the requirement along with the assertion of its correctness.

We would like to adopt the above scheme as a general scheme for reasoning under uncertainty. In the process of generalizing the above scheme to a scheme for reasoning under uncertainty we have to take into account the following remarks.

Remark 6.1. The knowledge of an agent in a scheme for reasoning under uncertainty is subjective and incomplete. In particular, an agent may not be able to distinguish among certain requirements(specifications, formulas etc.).

Remark 6.2. The local decomposition knowledge of an agent may also be uncertain and this knowledge may not be understood fully by other agents as the agents possess incomplete fragments of the knowledge about the world.

Remark 6.3. The leaf agents having an access to the inventory of elementary objects may be able to select objects which satisfy the requirements not cartainly but in an acceptable degree only.

Remark 6.4. Agents may be able to classify objects approximately only, in terms of their closeness to certain model objects (standards, logical values etc.).

Remark 6.5. The general form of an inference rule under uncertainty of an agent ag whose children are $ag_1, ag_2, .., ag_k$ is of the form

(D) if $(x_1, (\Phi_1, \epsilon_1)) \wedge (x_2, (\Phi_2, \epsilon_2)) \wedge ... \wedge (x_k, (\Phi_k, \epsilon_k))$ then

$$(o(x_1, x_2, .., x_k), (\Phi, \epsilon))$$

where x_1,x_2,...,x_k are objects submitted by, respectively, ag_1, ag_2,...,ag_k and (Φ_1, ϵ_1),...,(Φ_k, ϵ_k) are approximate specifications (formulas) at agents ag_1 ,..., ag_k, $o(x_1, ..., x_k)$ is the object produced by ag from $x_1, ..., x_k$ by means of an operation o and (Φ, ϵ) is the approximate specification at ag. The intended meaning of (D) is: if the agent ag_1 can submit an object x_1 satisfying the approximate specification (Φ_1, ϵ_1) and ... and the agent ag_k can submit an object x_k satisfying the approximate specification (Φ_k, ϵ_k) then ag can apply the operation o to assembly the object $x = o(x_1, ..., x_k)$ which satisfies the approximate specification (Φ, ϵ).

Remark 6.6. Problem specifications are issued by the external agent cag (the *customer agent*) in a language understandable to some agents in the scheme (in particular, to the root agent R). The specific form of the language depends on the particular synthesis process.

The object x synthesized by the scheme as an approximate solution to a requirement is evaluated by the agent cag with respect to its local knowledge. The process of learning the correct synthesis of solutions to a given specification is concluded when the two evaluations are consistent.

We would like to adopt rough mereology as a foundational basis for a general scheme for reasoning under uncertainty. We will therefore accept the following assumptions.

Assumption 1. Universes of objects (universes of discourse) of agents are models of L_{rm} in which certain collections of objects, called *standard objects*, are distinguished. The rough inclusions of the universes induce rough mereological distance functions in their respective domains by means of which objects are perceived and characterized with respect to the standards in the respective universe.

Assumption 2. The semantics of the approximate logic of formulas of the form (Φ, ϵ) of any agent ag is defined in terms of standards of ag and the rough mereological distance function in the universe of objects of the agent ag.

Assumption 3. Local uncertainty measures (uncertainty coefficients ϵ) are propagated from the children of an agent ag to the agent ag by means of functions extracted from intensional dependencies discovered from the knowledge (information systems) of agents.

Our approach can be precisely described as "an analytical approximate reasoning" in the sense that all necessary ingredients for the reasoning scheme are extracted or inferred from the empirical knowledge represented by the information systems of the agents.

6.2 The agent structure

We will discuss here the structure of a single agent ag in a scheme S of agents.

We begin with a theoretical result which will simplify our treatment of the reasoning scheme. By a *quasi-rough inclusion* μ_o in a set U we will understand a function

$$\mu_o : U \times U \rightarrow [0,1]$$

which satisfies the following conditions

(i) $\mu_o(x,x) = 1$;

(ii) if $\mu_o(x,y) = 1$ then $\mu_o(z,y) \geq \mu_o(z,x)$ for any $z \in U$;

(iii) $\mu_o(x,y) = \mu_o(y,x)$.

We recall that a *t-conorm* $\perp$ [11] is a function $\perp : [0,1]\times[0,1] \rightarrow [0,1]$ such that $\perp$ is increasing coordinate-wise, commutative, associative and $\perp(r,0) = r$. We extend the operators $\top, \perp$ over the empty set of arguments and over singletons by adopting the following convention : $\top(\emptyset) = 1, \perp(\emptyset) = 1, \top(r) = r = \perp(r)$. We observe that by the associativity and commutativity of $\top$ and $\perp$, the values $\top(x_1, ..., x_k)$ and $\bot(x_1, ..., x_k)$ are defined uniquely for any finite set of arguments. We have the following proposition, whose straightforward proof is omitted.

Proposition 15 *A quasi-rough inclusion μ_o on a set U can be extended to a rough inclusion on the power set exp(U); in particular, an extension of μ_o is defined by the following formula where $X, Y \subseteq U$*

$$\mu(X, Y) = \top\{\bot\{\mu_o(x, y) : y \in Y\} : x \in X\}.$$

$\square$

A pair (U, μ_o) where U is a set and μ_o is a quasi - rough inclusion on U is called a *quasi - model* of L_{rm}.

We consider an agent ag in the scheme. We will call the *label of the agent ag* the tuple

$$lab(ag) = (\mathsf{A}(ag), M(ag), L(ag), Link(ag), O(ag), St(ag),$$
$$Unc\_rel(ag), H(ag), Unc\_rule(ag), \; Dec\_rule(ag))$$

where

1. $\mathsf{A}(ag) = (U(ag), A(ag))$ is an information system of the agentag;

2. $M(ag) = (U(ag), [0, 1], F(ag))$ is a quasi-model of L_{rm} with a quasi-rough inclusion $F(ag)(\mu) = \mu_o(ag)$ in the universe $U(ag)$.

3. $L(ag)$ is a set of unary predicates in a predicate calculus interpreted in the set$U(ag)$;

4. $St(ag) = \{st(ag)_1, ..., st(ag)_n\} \subseteq U(ag)$ is the set of standard objects at ag;

5. $Link(ag)$ is a collection of strings of the form $ag_1ag_2...ag_kag$; the intended meaning of a string $ag_1ag_2...ag_kag$ is that $ag_1, ag_2, .., ag_k$ are children of ag in the sense that ag can assemble complex objects (constructs) from simpler objects sent by $ag_1, ag_2, ..., ag_k$. In general we can assume that for some agents ag we may have more than one element in $Link(ag)$ which represents the possibility of re-negotiating the synthesis scheme.

6. $O(ag)$ is the set of operations at ag; any $o \in O(ag)$ is a mapping from the cartesian product $U(ag_1) \times U(ag_2) \times ... \times U(ag_k)$ into the universe $U(ag)$ where $ag_1ag_2...ag_k \in Link(ag)$;

7. *Unc_rel(ag)* is the set of uncertainty relations unc_rel_i of type

$$(o_i, \rho_i, ag_1, ag_2, ..., ag_k, ag, \mu_o(ag_1), ..., \mu_o(ag_k), \mu_o(ag))$$

where $ag_1 ag_2 ... ag_k ag \in Link(ag)$ and ρ_i is such that

$$\rho_i((x_1, \epsilon_1), (x_2, \epsilon_2), ., (x_k, \epsilon_k), (x, \varepsilon))$$

holds for $x_1 \in U(ag_1), x_2 \in U(ag_2), .., x_k \in U(ag_k)$ and $\varepsilon_1, \varepsilon_2, .., \varepsilon_k \in [0,1]$ iff $\mu_o(x_j, st(ag_j)_i = \epsilon_j$ for $j = 1, 2, .., k$ and $\mu_o(x, st(ag)_i) = \epsilon$ for the collection of standards $st(ag_1)_i, st(ag_2)_i, ..\ ., st(ag_k)_i, st(ag)_i$ such that

$$o_i(st(ag_1)_i, st(ag_2)_i, .., st(ag_k)_i) = st(ag)_i.$$

Uncertainty relations express the agents knowledge about relationships among uncertainty coefficients of any agent ag and uncertainty coefficients of its children. The relational character of these dependencies expresses their intensionality.

8. $Unc\_rule(ag)$ is the set of uncertainty rules unc_rule_j of type

$$(o_j, f_j, \mu_o(ag_1), \mu_o(ag_2), ., \mu_o(ag_k), \mu_o(ag))$$

of the agent ag where $ag_1 ag_2 ... ag_k ag \in Link(ag)$ and $f_j : [0,1]^k \longrightarrow [0,1]$ is a function which has the property that there exists a collection of standards $st(ag_1), st(ag_2), ..., st(ag_k), st(ag)$ and

if objects $x_1 \in U(ag_1), x_2 \in U(ag_2), .., x_k \in U(ag_k)$

satisfy the conditions $\mu_o(x_i, st(ag_i)) \geq \varepsilon(ag_i)$ for $i = 1, 2, .., k$

and $\mu_o(o_j(x_1, x_2, ..., x_k), st(ag)) \geq \varepsilon(ag)$

then $f_j(\varepsilon(ag_1), \varepsilon(ag_2), .., \varepsilon(ag_k)) \geq \varepsilon(ag)$.

Uncertainty rules provide functional operators for propagating unceratinty measure values from the children of an agent to the agent; their application is in negotiation processes where they inform agents about plausible uncertainty bounds.

9. $H(ag)$ is a strategy which produces uncertainty rules from uncertainty relations; to this end, various rigorous formulas as well as various heuristics can be applied.

10. Dec-rule(ag) is a set of decomposition rules dec_rule_i of type

$$(o_i, \Phi(ag_1), \Phi(ag_2), .., \Phi(ag_k), \Phi(ag))$$

where $\Phi(ag_1) \in L(ag_1), \Phi(ag_2) \in L(ag_2), .., \Phi(ag_k) \in L(ag_k), \Phi(ag) \in L(ag)$ and $ag_1 ag_2 ... ag_k ag \in Link(ag)$ such that there exists a collection of standards $st(ag_1), st(ag_2), . ., st(ag_k)$, $st(ag)$ with the properties that

$$o_j(st(ag_1), st(ag_2), .., st(ag_k)) = st(ag),$$

$st(ag_i)$ satisfies $\Phi(ag_i)$ for $i = 1, 2, .., k$ and $st(ag)$ satisfies $\Phi(ag)$.

Decomposition rules are decomposition schemes in the sense they describe the standard $st(ag)$ and the standards $st(ag_1), ..., st(ag_k)$ from which the standard $st(ag)$ is assembled under o_i.

6.3 The approximate logic of an agent

We will comment briefly on the semantics of approximate formulas of the form (Φ, ε); our discussion is a very concise extract from [15] where the approximate logic L_{approx} of a system of agents is discussed formally. Consider a predicate $\Phi \in L(ag)$ and $\varepsilon \in [0, 1]$. The approximate formula (Φ, ε) has the intended meaning of a formula Φ satisfied in a degree ε; formally, we will say that a construct (object) $x \in U(ag)$ satisfies the approximate formula (Φ, ε) iff there exists a standard $st(ag)$ such that $st(ag)$ satisfies the formula Φ and $\mu_o(ag)(x, st(ag)) \geq \varepsilon$. In particular, for a decomposition rule dec_rule_i as in (10) above x satisfies $(\Phi(ag), \varepsilon)$ whenever $\mu_o(ag)(x, st(ag)) \geq \varepsilon$; clearly, $st(ag)$ satisfies the approximate formula $(\Phi(ag), 1)$.

6.4 The approximate reasoning by a system of agents

We now consider a system S of agents over an inventory INV. We assume that the relation $\leq$, defined by $ag' \leq ag$ iff $ag_1 ag_2 ... ag_k ag \in Link(ag)$ and there exists $i \leq k$ such that $ag' = ag_i$, orders S into a tree; we assume that any agent ag in S has exactly n standards which satisfy the composition rule in the sense that if $ag_1 ag_2 ... ag_k ag \in Link(ag)$ and $ag_1^i ag_2^i ... ag_{k_i}^i ag_i \in Link(ag_i)$ for $i = 1, 2, .., k$ then for any $j = 1, 2, .., n$ the composition

$$o_j(ag) \circ (o_j(ag_1), ..., o_j(ag_k))$$

produces from standards $st(ag_1^1)_j$,...., $st(ag_{k_k}^k)_j$ the standard $st(ag)_j$. We denote by the symbol $Root(S)$ the root agent of the scheme S and the symbol $Leaf(S)$ will denote the set of leaf (inventory) agents of S. We now present the procedure of approximate solution synthesis by the scheme S. The procedure involves two stages of communication/negotiation process: the top-down communication/negotiations process and the bottom-up communication process. We begin with the top-down stage described in the following steps.

Step 1. A requirement Φ in $L(\mathrm{Root}(S))$ is issued by the agent cag to the agent scheme.

Step 2. The root agent selects a standard $st(Root(S))_i$ such that $st(Root(S))_i$ satifies Φ (if this is not possible then the procedure halts).

Step 3. The agent $Root(S)$ selects an uncertainty coefficient $\epsilon(Root(S))$; the meaning of this choice is that if an object x satisfies the approximate formula $(\Phi^i(Root(S)), \epsilon(Root(S)))$ then the object x satisfies the requirement Φ.

Step 4. The agent $Root(S))$ communicates with its children

$$ag_1, ag_2, ..., ag_k$$

and negotiates with them the choice of uncertainty coefficients $\epsilon(ag_j)$, $j = 1, 2, ..., k$, such that

$$f_i(\epsilon(ag_1), \epsilon(ag_2), ..., \epsilon(ag_k)) \geq \epsilon(Root(S))$$

where f_i is the uncertainty function in the uncertainty rule unc_rule_i of $Root(S))$.

Step 5. The negotiation procedures are repeated by agents ag_1, ag_2,... and their children until leaf agents are reached. The result of the successful negotiation process is the set $\{\epsilon(ag)\}$ of uncertainty coefficients at all agents ag; at any agent ag the counterpart of the condition in Step 4 holds.This defines the set of specifications for all agents.

Step 6. Any leaf agent ag realizes its specification $(\Phi^i(ag), \epsilon(ag))$ by choosing an object $x \in INV \cap U(ag)$ which satisfies $(\Phi^i(ag), \epsilon(ag))$.

The bottom - up communication process consists in the sequence of steps of the following form.

Step 7. Any agent $ag \neq Root(S)$ sends to its parent the object x assembled from objects sent by its children and satisfying their specifications . The agent ag calculates and communicates to its parent the vector $[\mu_o(x, st(ag)_j : j = 1, 2, .., n]$ of distances of the object x from its standards. The leaf agents send the objects chosen in the inventory also with the vector of distances from standards.

Step 8. The agent $Root(S)$ assembles the final object x_{Φ}, calculates the vector of distances from its standards and checks that the object x_{Φ} satisfies the specification $(\Phi^{i}(Root(S)), \epsilon(Root(S)))$. In the positive case the object x_{Φ} is issued to the agentcag.

We also would like to underline the control mechanism incorporated into the above procedure.

Remark 6.7. Let us observe that the condition imposed on the uncertainty function f_i does not guarantee that when the children $ag_1, ag_2, ..., ag_k$ of ag select objects x_1, x_2,...,x_k such that

$$\mu_o(ag_j)(x_j, st(ag_j)_i) \geq \epsilon(ag_j)$$

then we have $\mu_o(ag)(o_i(x_1, .., x_k), st(ag)_i) \geq \epsilon(ag)$; such demand would be unrealistic. It may happen therefore that the agent ag will find that it cannot assenmble an object which would satisfy its specification. In this case it is able to interrupt the synthesis process and to demand better quality parts (e.g. greater uncertainty coefficients $\epsilon(ag_j)$) or a renewal of the negotiation process etc. Our scheme acts therefore as a controller.

We will comment briefly on the negotiation tasks in the scheme S.

Remark 6.8. We would like to observe that the agents in S can communicate by means of mereological decomposition schemes indued by their quasi-rough inclusions; the communication process is based on the observation that for any $ag_1 ag_2..ag_k ag \in Link(ag)$, the agent ag construct its complex objects as classes in the sense of mereology of Leśniewski of simpler objects and these simpler objects in turn are complex objects (classes in mereological sense) in the universes of children.

The reader will find in [24] an example of a negotiation process based on boolean reasoning [3], [44] and in [25], [28] an idea of a rough mereological controller.

7 Conclusions

We have presented a conceptual scheme for approximate reasoning about complex systems in the processes of synthesis of complex systems from simpler parts. Our analysis is applicable as well to problems of design, analysis and control in complex systems [31], [40]. Our approach is based on rough mereology and rough inclusions determine local decomposition schemes of agents by means of which agents establish the mereological hierarchies of objects. The relationships among agents resulting from the inferred local decomposition schemes are encoded in strings in *Link* and they are used in determining a scheme formation for synthesis of an approximate solution to a given requirement. All relations and functions which determine the mechanisms for propagation of uncertainty as well as mechanisms of negotiatons are inferred from knowledge of agents represented in their information systems. The adaptiveness of our scheme is achieved by means of of an adjustment of rough inclusions chosen by agents, modifications of uncertainty relations and uncertainty functions due to the appearance of new yet unseen objects (constructs), improvements in the performance of agents due to the learning processes, and possibilities for redesigning the scheme due to new mereological hierarchies that result from other changes to the scheme and its environment. The limitations of this scheme are due to the complexity of learning tasks leading to the choice of rough inclusions, uncertainty relations and rules, and to the scheme formation.

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