Ontology and the Logistic Analysis of Reality[1]

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From N. Guarino and R. Poli (eds.), *Proceedings of the International Workshop on Formal Ontology in Conceptual Analysis and Knowledge Representation,* Padova: Institute for Systems Theory and Biomedical Engineering of the Italian National Research Council, 1993, 51–68.

§ 1. Introduction

I shall attempt in what follows to show how mereology, taken together with certain topological notions, can yield the basis for future investigations in formal ontology.[2] I shall attempt to show also how the mereological framework here advanced can allow the direct and natural formulation of a series of theses – for example pertaining to the concept of *boundary* – which can be formulated only indirectly (if at all) in set-theoretic terms.

Already Whitehead employed the instruments of mereology and topology as the basis of a formal ontology, though in his case this ontology was restricted to events.[3] The long–term aim of the present framework, in contrast, is to serve as a basis for a formal ontology of the common-sense world, a world which includes at least the following structures and dimensions:

space (is located at, is at, is in)

things (organs, bodies, institutions)

time (occurs at, exists at)

events, processes, states (moves through, moves from, changes, begins, ends)

qualities (red, hot)

species at different levels of generality (cat, molecule, run, sentence, salute)

matter, stuff, mixtures (gold, water, air).

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[1] The present paper was prepared as part of the project "Formalontologische Grundlagen der künstlichen Intelligenzforschung" sponsored by the Swiss National Foundation for Scientific Research, 1991-93. I am grateful to Roberto Casati and to Wojciech Żełaniec for valuable comments. [An early version appeared in an unpublished Festschrfit for Guido Küng *Analytic* Phenomenology, edited by G. Haefliger and P. M. Simons (eds.).]

[2] Thus following in the footsteps of *inter alia* Menger 1940, Tarski 1956, Grzegorczyk 1977 and Section 6 of Tiles 1981. See also the material that is gathered in Smith, ed., 1982, as also Simons 1989, Libardi 1990 and Fine 1993.

[3] See Whitehead 1929 and also Clarke 1981, 1985.

Our investigations are thus to be seen as part of the project of a formal ontology that has been advanced of late in the field of artificial intelligence, for example in Hayes 1985.[4]

We shall be concerned, like the mathematician, to produce formally precise theories of structures of certain sorts, and in such a way that it is the structures themselves that hold our interest and not the formal machinery that has been set up to describe them. Hence our axioms will be chosen primarily for the sake of the light they throw on their intended subject-matters (and not, e.g., from the perspective of logical independence). Our concerns differ from those of the mathematician in that we shall deal not with abstract structures for their own sake, but rather (as far as possible) with certain specific sorts of naturally occurring structures in the real, spatio-temporal world. Like Frege, Russell and Lesniewski, we shall evince no 'semantic' or 'model-theoretic' concerns. The world itself is the only model in which we take a serious interest, and we shall attempt to do justice to this world as it is given in ordinary experience. Thus for example we shall formulate a system which will enable us to prove that every boundary is the boundary of something, that there are no scattered ojbects (in addition to the non-scattered parts), and so on.

§2. Formal Machinery[5]

Classical first order logic with identity and descriptions will be assumed without ceremony.[6] Variables x, y, z, etc. will range over entities (particulars, individuals) in general. Here the term 'entity' is to be understood as ranging over realia of all sorts. Our quantifiers are otherwise unrestricted, embracing inter alia Roderick Chisholm's left foot and the exterior surface of my house; my present headache and the three-dimensionally extended colour of this green glass cube. They embrace what is continuous or discontinuous, connected or non-connected; and they embrace also volumes of space and intervals of time, as well as the parts and moments of all of these entities.

Task: to find a semantics that is both ontologically and cognitively adequate. Set theory is neither.

Constituency

Everything continuous is such that we are able to discern within it two basic sorts of constituent or part: boundaries and interior parts. We shall attempt to use simple mereological and topological notions in order to supply a more precise statement of this intuitive idea.

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[4] There, however, it has been set–theoretical instruments which have hitherto prevailed. (See e.g. Davis 1990, p. 248.) Those working in formal ontology in the A.I. field have in addition neglected the historical antecedents of their work, which go back at least as far as Whitehead 1906. Küng 1963 can in some respects be regarded as the first definitive survey of the field.

[5] What follows is based on material published as Smith 199*x* and (revised) 199xa, which also contain proofs of the theorems stated below. I am grateful to Roberto Casati, Carola Eschenbach, Reinhardt Kleinknecht, Achille Varzi, Graham White and Wojciech Żełaniec for valuable comments.

[6] In a complete account we should have to find some means to take account of the fact that the term-forming operator 'σ' introduced below is not always defined.

The two primitive non-logical notions adopted in what follows are: is a constituent of and is an interior part of. The first of these notions is purely mereological in the sense of Leśniewski. The second represents a move from mereology into the province of topology.

We say x is a **constituent** of y, and write 'x C y', when x is any sort of part of y, including an improper part (so $x \mathrm{C} y$ will be consistent with x's being identical to y). Three further purely mereological notions can be defined immediately:

DC1	x **overlaps** y:	$x \mathrm{O} y := \exists z (z \mathrm{C} x \wedge z \mathrm{C} y)$
DC2	x is **discrete** from y:	$x \mathrm{D} y := \neg x \mathrm{O} y$
DC3	x is a **point**:	$\mathrm{Pt}(x) := \forall y (y \mathrm{C} x \rightarrow y = x)$

As axioms governing C we shall assume the universal closures of:

AC1 $\quad x \mathrm{C} y \leftrightarrow \forall z (z \mathrm{O} x \rightarrow z \mathrm{O} y)$

AC2 $\quad x \mathrm{C} y \wedge y \mathrm{C} x . \rightarrow x = y$

(Generally speaking we suppress all initial universal quantifiers in our statements of axioms and theorems.) From **AC1** and **AC2** and the usual axioms of identity it follows that our system of mereology is extensional, and in particular that $x = y \leftrightarrow \forall z (z \mathrm{C} x \leftrightarrow z \mathrm{C} y)$. From **AC1** it follows also that

TC1 $\quad x \mathrm{C} x \qquad$ C is reflexive

and

TC2 $\quad x \mathrm{C} y \wedge y \mathrm{C} z . \rightarrow x \mathrm{C} z \qquad$ C is transitive

We say that a condition 'ϕ' in a single free variable 'x' is **satisfied** iff the sentence 'ϕx' is true for at least one value of 'x'. Intuitively we are to suppose that each satisfied condition 'ϕ' picks out a certain unique entity, the **sum** (fusion or join) of all those entities in the world which ϕ, an entity which we shall represent by '$\sigma x(\phi x)$'. The sum of ϕers is to be distinguished from the extension of the concept ϕ: not everything that is in the sum of ϕers need itself be such as to ϕ (thus my leg is in the sum of Britons, but it is not itself a Briton).

The sum of ϕers can be defined as that entity y which is such that, given any entity w, w overlaps with y if and only if w overlaps with something that ϕs. That is:

DC4 $\quad \sigma x(\phi x) := \iota y (\forall w (w \mathrm{O} y \leftrightarrow \exists v (\phi v \wedge w \mathrm{O} v)))$

We can then prove

TC3 $\quad y = \sigma x(\phi x) \rightarrow \forall x (\phi x \rightarrow x \mathrm{C} y)$

Empty sums do not exist (they are not a part of reality). Thus if ϕ is a non-satisfied condition, then '$\sigma x(\phi x)$' is undefined. The uniqueness of sums, where they are defined, is guaranteed by **AC1**. We stipulate further that:

AC3 $\exists x\ \phi x \rightarrow \exists y\ \forall w\ (w\ O\ y \leftrightarrow \exists v\ (\phi v \wedge w\ O\ v))$,

which guarantees the existence of sums for satisfied conditions. **AC1-3** define a system equivalent to classical extensional mereology as defined in Simons 1987.

From the usual axioms for identity we have $\exists x\ (x = x)$, from which we can prove a theorem to the effect that the universe exists:

TC4 $\exists x\ \forall y\ (y\ C\ x)$

Further:

TC5 $y\ C\ \sigma x(\phi x) \leftrightarrow \forall w\ (w\ C\ y \rightarrow \exists v\ (\phi v \wedge w\ O\ v))$

y is a constituent of the sum of ϕers if and only if all constituents of y overlap with some ϕer.

We have already noted that not all constituents of the whole $\sigma x(\phi x)$ need be such as to ϕ. When $y\ C\ \sigma x(\phi x)$ iff ϕy, then we say that ϕ is a **distributive** condition, and we can prove that $\phi(\sigma x(\phi x))$. Examples of distributive conditions are (for some fixed entity t): is a constituent of t, is a boundary of t, and is an interior part of t.[7]

We can prove further a theorem to the effect that we can form arbitrary finite unions in the following sense:

TC6 $\exists z\ \forall w\ (z\ O\ w \leftrightarrow .\ z\ O\ x \vee z\ O\ y)$

We define:

1	$:= \sigma x(x = x)$	universe
$x \cup y$	$:= \sigma z(z\ C\ x \vee z\ C\ y)$	union
$x \cap y$	$:= \sigma z(z\ C\ x \wedge z\ C\ y)$	intersection
$\bigcap_{\phi}$	$:= \sigma x(\forall y\ (\phi y \rightarrow x\ C\ y))$	intersection of ϕers
x'	$:= \sigma z(z\ D\ x)$	complement
$x - y$	$:= \sigma z(z\ C\ x \wedge z\ D\ y)$	difference

Note that all set-theoretical associations of these terms are to be resolutely suppressed. Note, also, that intersections, complements and differences are not always defined. We can in fact prove

TC7 $x\ C\ y \wedge x \neq y. \rightarrow \exists z\ (z = y - x)$ remainder principle

There are reasons for rejecting this principle, as also the principle **TC6**, when the range of our variables is restricted in certain ways, though to this end we would have to weaken the

[7] Cf. Bunt's definition of downward homogeneous properties (1979, p. 269).

axioms **AC1** and **AC2** on which the proof of these principles hinges. (To see why one might reject the remainder principle, suppose that the range of our variables is restricted to bona fide material things, and suppose further that the human body and the human heart are both material things in this sense; then it is not clear that the entity which results from detaching heart from body mereologically is also a material thing in the same sense.)

Interior Parts

The notion of interior part can be elucidated, roughly, as follows. Some entities are what we might call tangential to, i.e. such as to touch or cross the boundaries of, other entities. Some entities are themselves boundaries of other entities, though we note that the boundary of an entity may be outside the entity it bounds (as for example in the case of a hole, or an open interval in the real line). When x is a constituent of y that is off – which is to say: shares no parts in common with – the boundary of y, i.e. is neither tangential nor itself a boundary, we say that x is an **interior part** of y and write 'x P y'. We then stipulate:

AP1	$x \mathrm{P} y \rightarrow x \mathrm{C} y$	P is a special sort of C
AP2a	$x \mathrm{P} y \wedge y \mathrm{C} \mathrm{z}. \rightarrow x \mathrm{P} \mathrm{z}$	left monotonicity
AP2b	$x \mathrm{C} y \wedge y \mathrm{P} \mathrm{z}. \rightarrow x \mathrm{P} \mathrm{z}$	right monotonicity
AP3	$x \mathrm{P} y \wedge x \mathrm{P} \mathrm{z} .\neg x \mathrm{P} (y \cap \mathrm{z})$	condition on finite intersections
AP4	$\forall \mathrm{x}(\phi x \neg x \mathrm{P} y) \neg \sigma \mathrm{x}(\phi x) \mathrm{P} y$	condition on arbitrary unions
AP5	$\exists \mathrm{y}(x \mathrm{P} y)$	
AP6	$x \mathrm{P} y \neg x \mathrm{P} \sigma t(t \mathrm{P} y)$	

all of which follow from the usual axioms for a topological space.

AP5 is very strong, and allows us to infer an initially counterintuitive-seeming theorem to the effect that the universe is an interior part of itself:

TP1 $\mathbf{1} \mathrm{P} \mathbf{1}$.

The universe is, as we might also say, 'unbounded'. (What, indeed, would the boundaries of the universe be, if 'boundary' is understood in the common-sense way as that which separates e.g. an apple from its surroundings?) Indeed we can prove that:

TP2 $\forall x (x \mathrm{P} \mathbf{1})$.

Every entity is an interior part of the universe.
From **AP4** it follows that P determines a distributive condition, i.e. that

TP3 $t \mathrm{C} \sigma \mathrm{x}(x \mathrm{P} y) \leftrightarrow t \mathrm{P} \mathrm{y}$.

Hence also we have

$$t \text{ C } \sigma x(x \text{ P } y) \leftrightarrow t \text{ P } \sigma x(x \text{ P } y)$$

and:

TP4 $\sigma x(x \text{ P } y) \text{ P } y$.

Boundaries

As a first step towards defining what it is for x to be a boundary of y, we define 'x x y' (x **crosses** y) by:

DC5 $x \text{ X } y := \neg\, x \text{ C } y \wedge \neg\, x \text{ D } y$

or, equivalently, for $y \neq \mathbf{1}$,

$$x\, X\, y := x \text{ O } y \wedge x \text{ O } (\mathbf{1} - y),$$

i.e., x overlaps both y and its complement. From this it follows trivially that no entity crosses itself and that the universe crosses every entity not identical with the universe itself. We now define 'x St y' (x **straddles** y) by:

DP1 $x \text{ St } y := \forall z\, (x \text{ P } z \rightarrow z \text{ X } y)$.

An entity x straddles an entity y whenever x is such that everything of which it is an interior part crosses y. The definitions then yield immediately that x St y Ø ¬ x P y, from which we can prove:

TP5 $x \text{ C } y \rightarrow .\, x \text{ P } y \vee x \text{ St } y$

Every constituent of y is either an interior part of y or such as to straddle y. This follows from **AP1**, **AP2a** and definitions. As a theorem we also have: $\neg x \text{ P } x \rightarrow x \text{ St } x$.

The entities straddling a given entity can be divided, intuitively, into two classes. On the one hand are those which include among their constituents a boundary of the straddled entity. On the other hand there are those – characteristically non-connected – which include no such boundary. We shall refer to the first group as **tangents**. As an example of a non-tangential entity straddling y, consider the sum of two points, both off the boundary of some three-dimensional solid y, one of which is interior to y, the other exterior. If we examine case V, where x is not merely such as to straddle y but is in fact a boundary of y, then we see that what is characteristic of this case is that here x is such that not merely it but also all its constituents are such as to straddle the bounded entity.[8] Accordingly we can define **boundary** as follows:

[8]Cf. Chisholm 1989, ch. 8, "Boundaries".

DP2 $x \text{ B } y := \forall z\, (z \text{ C } x \rightarrow z \text{ St } y)$.[9]

We can now define what it is for x to be a **tangent** of y:

DP3 $x \text{ T } y := \exists z\, (z \text{ C } x \wedge z \text{ B } y)$

i.e a tangent of y is any entity which has as part a boundary of y. From this definition we can prove that tangents are straddlers, and also that every boundary of y is a tangent of y and is thereby also not an interior part of y. We can prove further, by inspection of the definitions, that

TB1 $x \text{ B } y \leftrightarrow \forall z\, (z \text{ C } x \rightarrow z \text{ T } y)$,

so that, as required, all constituents of boundaries of y are not merely straddlers but in fact tangents of y.

Closure
We can prove further:

TB2 $x \text{ B } y \wedge y \text{ B } z. \rightarrow x \text{ B } z$ transitivity

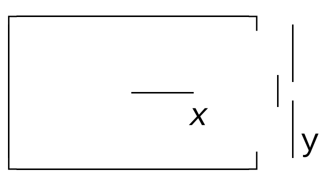
We also have:

TB3 $x \text{ C } y \wedge y \text{ B } z. \rightarrow x \text{ B } z$
TB4 $x \text{ T } (y \cup z) \rightarrow. x \text{ T } y \vee x \text{ T } z$ splitting

We can prove also the following collection principle for boundaries:

TB5 $\forall x\, (\phi x \rightarrow x \text{ B } y) \rightarrow \sigma x(\phi x) \text{ B } y$

[9]This concept of boundary is unintuitive in one respect, in that it comprehends also *slits*:



where $\neg\, x \text{ C } y$. It seems to be an important mark of the pre-analytic concept of boundary, however, that a boundary of y should divide y from something else. Considerations along these lines may be met by setting:

$$x \text{ B } y := \forall z\, (z \text{ C } x \rightarrow. z \text{ St } y \wedge z \text{ St } (1 - (y \cup x))$$

Topology

These theorems enable us to show that the system so far established defines a topological space in the standard sense, by defining the closure cl(x)of $x \neq \mathbf{1}$ as the union of x with all its boundaries:

DP4 cl(x):= $x \cup \sigma y(y \text{ B } x)$

Closure thus defined satisfies the usual Kuratowski axioms:

I. x C cl(x)
II. cl(cl(x)) = cl(x)
III. cl($x \cup y$) = cl(x)$\cup$ cl(y)

An entity is called **open**if it is identical with the sum of its interior parts, **closed** iff it is identical with its closure. cl(x)as defined above can be shown to be identical to the standard topological closure defined equivalently as the union of x with its accumulation points (see below) or as the intersection of all closed entities containing x. Except that cl( ) is a partial function, and cl(**1**), in particular, is not defined. Thus the univers, while open, is not closed (in contrast to what is the case in standard topology). A **dense** entity, standardly, is an entity x for which cl(x)= **1**.

The **maximal boundary** of x, defined by

DP5 bdy(x):= $\sigma y(y \text{ B } x)$,

now corresponds to the standard topological boundary, defined as the intersection of the closure of an entity with the closure of its complement. Further our **interior**, defined by

DP6 int(x):= $\sigma y(y \text{ P } x)$,

corresponds to the standard topological interior, defined as the difference between an entity and its boundary. Both bdy( ) and int( ) are partial functions.

An entity is called **open** iff it is identical with its interior. From this we can prove that an entity is open if and only if its complement is closed (use **TC5**). Moreover

TT1a Every finite intersection of open entities is open.

And further:

TT1b Every arbitrary union of open entities is open.

Similarly, and no less familiarly, we can prove:

TT2a Every finite union of closed entities is closed.
TT2b Every arbitrary intersection of closed entities is closed.

Either **TT1a** and **TT1b** or **TT2a** and **TT2b** can in turn be shown to be themselves sufficient to serve as axioms for a topological space.

We could also however take 'cl' as primitive and set, for $x \neq \mathbf{1}$:

$$\mathrm{bdy}(x) := \mathrm{cl}(x) \cap (\mathrm{cl}(\mathbf{1} - x))$$
$$\mathrm{int}(x) := x - \mathrm{bdy}(\mathrm{x}).$$

If we then define:

$$x \mathrm{~P~} y := x \mathrm{~C~} \mathrm{int}(y)$$
$$x \mathrm{~B~} y := x \mathrm{~C~} \mathrm{bdy}(y)$$

then from the Kuratowski axioms and the axioms for 'C' we can prove the axioms **AP1–6** set out above.

Dependent Existence and the Brentanian Theses

The remarks above are non-controversial reformulations of standard topological ideas on a mereological basis. Now, however, we wish to go further and capture mathematically certain ontological intuitions pertaining to ordinary material objects, extended in three-dimensional space and enjoying qualities of for example shape and colour. We wish to capture, if one will, the mathematical structures characteristic of the common-sense world. Three layers of such intuitions can be distinguished:

1. the layer corresponding to general topological notions of boundary, interior, etc., which has been treated above;

2. the layer corresponding to the general properties of three-dimensional space as we conceive it; this space is 'real' in the sense that it is not an abstract construction; thus it allows no space-filling curves, no objects of fractional dimension, etc.

3. the layer corresponding to the special topological properties of material objects and their associated qualities.

What follows is a provisional attempt to formulate some of the principles underlying 3. It is provisional not least because the definitive statement of such principles must await a more adequate understanding of the general properties of space.

Intuitively, we wish it to be the case that every entity smaller than the universe has a boundary:

AB1 $\mathrm{y} \neq \mathbf{1} \rightarrow \exists x\,(x \mathrm{~B~} y).$

This does not imply that the only open entity is **1**. Rather, it tells us that every open entity smaller than the universe is bounded, as it were, on at least one side or in at least one place (consider the case of the Western hemisphere of the universe or of the interstellar vacuum). The boundary itself need then not necessarily be a constituent of the entity bounded, and indeed that this should be the case in general is ruled out by:

TP6 $x \mathrm{~B~} y \wedge y \mathrm{~P~z} . \rightarrow x \mathrm{~B~} (\mathrm{z} - y)$

From this and **TP2** it follows in particular that every boundary of *y* is also a boundary of the complement of y.

From **TP6** it follows trivially that

$$x \mathrm{~B~} x \wedge x \mathrm{~P~} y . \rightarrow x \mathrm{~B~} (\mathrm{y} - x).$$

Imagine *x* is a point in the interior of a three-dimensional solid y. Then $y - x$ is here the result of deleting this point in such a way as to produce an entity which has a non-constituent boundary within its interior.[10] The opposition between exterior and interior boundaries will receive more detailed attention in what follows.

From **TP6** and **TP1** it follows no less trivially that

TP7 $x \mathrm{~B~} x \rightarrow x \mathrm{~B~} (\mathbf{1} - x)$,

whence also we can infer that, for any *x*, $\sigma \mathrm{y}(x \mathrm{~B~} y) = \mathbf{1}$, whence also we can infer that B does not define a distributive condition in the first argument.

We can prove further that an entity *x* is self-bounding (i.e. that $x \mathrm{~B~} x$) if and only if it has no interior parts:

TP8 $x \mathrm{~B~} x \leftrightarrow \neg\exists t(t \mathrm{~P~} x)$

We can now prove that every boundary which is a constituent of that which it bounds is also self-bounding:

TP9 $x \mathrm{~B~} y \wedge x \mathrm{~C~} y . \rightarrow x \mathrm{~B~} x.$

which stands in conflict with the commonsensical intuition to the effect that that which bounds e.g. a surface is the outer form or edge of the surface. That the surface is self-bounding is consistent with its having as boundary in addition some proper part of itself, including its outer form.

We shall shortly be in a position to prove that every boundary is a boundary of itself. (Note, in this connection, that we do not have in general $x \mathrm{~B~} y \rightarrow x \mathrm{~B~} (x \cup y)$, from which we

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[10]Compare the discussion of 'slits' above.

could immediately infer that x B $y \rightarrow x$ B x by **TP9**. For consider, again, the case where x is some interior point of a solid z, and y = z – x. Then x B y, but it is not the case that x B z, because z, by hypothesis, has no boundaries within its interior.)

From **AB1** and **TP8** we could then prove that boundaries have no interior parts.
From **TB5** we can prove:

TP10 $\quad x \text{ B z} \wedge y \text{ B z} .\rightarrow (x \cup y) \text{ B z}$

And we have also:

TP11 $\quad x \text{ C } y \rightarrow . \, x \text{ B } y \vee x \text{ P } y \vee \exists \text{uv} (\text{u B } y \wedge \text{v P } y \wedge \text{u} \cup \text{v} = x)$

Every constituent part is either a boundary or an interior part or the union of a boundary and an interior part (where the disjunctions are of course exclusive).

Variants of Brentano's Thesis

We wish now to capture the commonsensical intuition to the effect that boundaries exist only as boundaries, i.e. that boundaries are dependent particulars: entities which are such that, as a matter of necessity, they do not exist independently of the entities they bound.[11] This thesis – which stands opposed to the set-theoretic conception of boundaries as, effectively, sets of points, each one of which can exist though all around it be annihilated – has a number of possible interpretations. One general statement of the thesis would assert that the existence of any boundary is such as to imply the existence of some entity of higher dimension which it bounds. Here, though, we may content ourselves with a simpler thesis, to the effect that every boundary is such that we can find an entity which it bounds of which it is a constituent and which is such as to have interior parts. Define first of all the predicate is a boundary by means of:

DP7 $\quad \text{Bd}(x) := \exists \text{y}(x \text{ B } y)$

We can then write:

AB2 $\quad \text{Bd}(x) \rightarrow \exists \text{zt} (x \text{ B z} \wedge x \text{ C z} \wedge \text{t P z})$ First Brentanian Thesis

From this the theorem to the effect that all boundaries are self-bounding can be inferred immediately via **TP9**. **AB2** is not very strong, however. For it seems that we have x B $y \rightarrow x$ B (y ∪ t) for any arbitrary t that is separate from the closure of y. Thus **AB2** is satisfied by choosing t such that t P t and setting z equal to the scattered object $x \cup$ t.

A Brentanian thesis of the required strength must impose on z in **AB2** at least the additional requirement of connectedness. To this end we define, for $x \neq \mathbf{1}$ and $y \neq \mathbf{1}$:

[11] Brentano 1976, Part One; Chisholm 1984; Smith 1992.

DCn1 $x \text{ S } y := \text{cl}(x)\text{D } y \wedge x \text{ D cl}(y)$

We then say that $\mathbf{1} - (x \cup y)$ **separates** x from y. Thus bdy(x)separates int(x)from int(**1**–x) in the given sense. We can then prove:

TS1 $x \text{ S } y \wedge \text{w C } x \wedge \text{v C } y \,.\!\rightarrow \text{w S v}$

Further we know that disjoint entities are separate if either both are open or both are closed.
Define **connected**:

DCn2 $\text{Cn}(x) := x \neq \mathbf{1} \wedge \neg\, \exists\text{yz (y S z} \wedge x = y \cup \text{z)}.$

We then have a new Brentanian thesis affirming, for connected boundaries, the existence of connected wholes which they are the boundaries of:

AB3 $\text{Bd}(x) \wedge \text{Cn(x)}. \rightarrow \exists\text{zt } (x \text{ C z} \wedge x \text{ B z} \wedge \text{Cn(z)} \wedge \text{t P z})$

Second Brentanian Thesis

Note that **DP2** yields no guarantee that boundaries are connected in the sense here defined.

Exterior and Interior Boundaries

Intuitively, boundaries can be divided into exterior and interior.[12] The exterior boundaries of x are, as it were, boundaries which separate x from the remainder of the universe. Exterior boundaries in this sense may or may not be constituents of the things (or other entities) they bound, and they may or may not be on the exterior of the relevant entity in the normal understanding of this phrase.[13] We can distinguish also however **interior boundaries** – the boundaries which would result, intuitively, if interior parts of x were exposed to the light of day by annihilation of what stands between them and x's exterior. Interior boundaries are in this sense potential boundaries; they are those constituents of x which are boundaries of interior parts of x but not themselves boundaries of x in the strict sense. We define:

DIB1 $x \text{ IB } y := x \text{ P } y \wedge x \text{ B } x$

We might consider also in this connection the idea of a slicing principle to the effect that, in those cases where $x \text{ B } y$ results from the fact that x is a deleted region inside some $\text{z} = y - x$, we can slice z along x to produce one or more entities of which x is both exterior boundary and constituent.

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[12]See Brentano 1976, Part One, I; Smith 1992.

[13]They may be the boundaries of holes, including internal cavities; see, on the wealth of possibilities in this respect, Casati and Varzi 1994.

Points

We can prove:

TPt1 $\forall y\, (y \text{ B } x \leftrightarrow x = y) \rightarrow \text{Pt(x)}$

A point is that which has no parts other than itself (**DC3**). We can now stipulate that a point has no boundaries other than itself (a condition which might also have been used as a definition of 'point'):

APt1 $\text{Pt}(x) \rightarrow \forall y\, (y \text{ B } x \leftrightarrow x = y)$

This is equivalent to the proposition:

$$\text{Pt}(x) \rightarrow x = \text{cl(x)},$$

which is one (mereological) formulation of the usual condition on a T_1 topological space. A more standard formulation would be:

$$\neg\forall x\ \forall \text{y}\ (x \neq y \wedge \text{Pt}(x) \wedge \text{Pt}(y).\ \rightarrow \exists \text{z}\ ((x \text{ P z} \wedge \neg \text{y P z}) \vee (\text{y P z} \wedge \neg x \text{ P z})))$$

From **APt1** it follows further that:

TPt2 $\text{Pt}(x) \wedge x \text{ B } y \wedge x \neq \text{y}.\ \rightarrow \neg\text{Pt}(y)$

and, by setting $y = \mathbf{1} - x$:

TPt3 $\text{Pt}(x) \rightarrow \exists y\, (x \neq y \wedge x \text{ B } y).$

This goes some way towards capturing the anti-set-theoretical intution to the effect that there are, in reality, no isolated points, though its limits are clear from the fact that it is consistent with the thesis that the universe as a whole is an isolated point.

A **neighbourhood** of a point x is any entity y of which x is an interior part. A punctured neighbourhood of x is a neighbourhood with x deleted. An **accumulation point** may now be defined as follows:

DA1 $x \text{ A } y := \text{Pt}(x) \wedge \forall \text{z}\ (x \text{ P z} \wedge x \neq \text{z}.\ \rightarrow (\text{z} - x) \text{ O } y)$

i.e., an accumulation point of y is any point x which is such that any punctured neighbourhood of x overlaps y.

We now prove:

$$\text{y is closed} \rightarrow \sigma\text{x}(x \text{ A } y) \text{ C y}$$

From the definitions we can prove

TPt4 $x \text{ A } y \rightarrow . \; x \text{ B } y \vee x \text{ P y}.$

By **TP11** we can prove generally that:

$$\text{Pt}(x) \wedge x \text{ C } y \; . \rightarrow x \text{ B } y \vee x \text{ P y}.$$

TPt5 $x \text{ B } y \wedge x \text{ D } y \wedge \text{Pt(x)}. \rightarrow x \text{ A y}$

We may now go on to define **interior points** and **boundary points** as follows:

DPt1 $x \text{ IPt } y := \text{Pt}(x) \wedge x \text{ P y}$
DPt2 $x \text{ BPt } y := \text{Pt}(x) \wedge x \text{ B y}$

Using axiom **AP3** we can prove further that interior points are accumulation points.

TPt6 $x \text{ IPt } y \rightarrow x \text{ A y}$

Exploiting an analogy with Brentano's notion of the 'full plerosis of an internal boundary'[14] we may define further:

DA2 $x \text{ FA } y := \text{Pt}(x) \wedge \forall \text{z} \, (x \text{ B z} \wedge x \neq \text{z}. \rightarrow \exists \text{t} \, (\text{t P } y \wedge \text{t C z} \wedge x \text{ A t}))$

x is a **full accumulation point** for *y* iff it is an accumulation point to *y* in all the directions in which *x* can serve as boundary (*x* is, as it were, the centre of a spherical ball within *y*).

TPt7 $x \text{ FA } y \rightarrow x \text{ A y.(xPu} \rightarrow \neg\text{(u–x)Dt)).}$

Things
Return, once again, to the Second Brentanian Thesis:

AB3 $\text{Bd}(x) \wedge \text{Cn}(x). \rightarrow \exists \text{zt} \, (x \text{ C z} \wedge x \text{ B z} \wedge \text{Cn(z)} \wedge \text{t P z}).$

This is still too weak if we wish to capture the intuition to the effect that boundaries in the real material world are boundaries of things. For we require at least the further requirement to the effect that the entity z in question is the object bounded and not its complement. By **TP6** each boundary behaves symmetrically in relation to the object and its complement. From the perspective of common sense, however, the boundary (of, say, this stone) is much more intrinsically connected to the stone than it is to the rest of the universe. To capture this

[14] Brentano 1976, Part One, I.

notion formally would require (what we do not yet have) an adequate formal account of things, which we can characterize briefly as three-dimensional material entities which are at the same time maximally connected. Thus my arm is three-dimensional and material but it is not a thing, and similarly the scattered whole consisting of my arm and this pen is three-dimensional and material but it, too, is not a thing.[15] To this end, and in conclusion, we shall define the notion of a 'component' or maximally connected entity. For values of x such that $\mathrm{Cn}(x)$we set:

DCn3 $\mathrm{cm}(x) := \sigma y(x \mathrm{C} y \wedge \mathrm{Cn}(y))$

The component of x is the maximal connected entity containing x.

We can then prove:

TCn1 $z = \mathrm{cm}(x) \rightarrow \forall y(\mathrm{Cn}(y) \wedge z \mathrm{C} y \,.\!\rightarrow y = z)$

Components are, if one will, those natural units from out of which the world is built.[16] Such natural units can be found not only in the realm of three-dimensional materal things, but also e.g. in the temporal dimension (salutes, weddings, lives, are natural units in the realm of events and processes). To deal with these matters, here, however, as also with the concepts of dimension (edge, surface) and with the relations between natural units and their underlying stuffs, all of this would lead us too far.

Bibliography

Brentano, Franz 1976 *Philosophische Untersuchungen zu Raum, Zeit und Kontinuum*, ed. by R. M. Chisholm and S. Körner, Hamburg: Meiner, Eng. trans., *Philosophical Investigations on Space, Time and the Continuum*, London: Croom Helm, 1988.

Bunt, H. 1979 "Ensembles and the Formal Semantic Properties of Mass Terms", in F. J. Pelletier, ed., Mass Terms. Some Philosophical Problems, Dordrecht: Reidel.

Casati, R. and Varzi, A. 1994 *Holes and Other Superficialities*, Cambridge, Mass.: MIT Press.

Chisholm, R. 1989 *On Metaphysics*, Minneapolis: University of Minnesota Press.

Chisholm, R. M. 1984 "Boundaries as Dependent Particulars", *Grazer Philosophische Studien*, 10, 87-95.

[15]See Smith 1992.

[16]See Smith 1991.

Clarke, B. L. 1981 "A Calculus of Individuals Based on *Connection*", *Notre Dame Journal of Formal Logic*, 22, 204-18.

Clarke, B. L. 1985 "Individuals and Points", *Notre Dame Journal of Formal Logic*, 26, 61-75.

Davis, E. 1990 *Representations of Commonsense Knowledge*, San Mateo, CA: Morgan Kaufmann.

Fine, Kit 1993 "Husserl's Theory of Part and Whole", in B. Smith and D. W. Smith, eds., *Cambridge Companion to Husserl*, Cambridge and New York: Cambridge University Press.

Grzegorczyk, A. 1977 "On Certain Formal Consequences of Reism", *Dialectics and Humanism*, 1, 75-80.

Hayes, Patrick J. 1985 "The Second Naive Physics Manifesto", in Hobbs and Moore, eds., 1-36.

Hobbs, J. R. and Moore, R. C. eds. 1985 *Formal Theories of the Common-sense World*, Norwood: Ablex.

Küng, Guido 1963 *Ontologie und die logistische Analyse der Sprache*, Vienna: Springer, Eng. trans. as *Ontology and the Logistic Analysis of Language*, Dordrecht: Reidel, 1967.

Libardi, Massimo 1990 *Teorie delle parti e dell intero. Mereologie estensionali*, Trento: Centro Studi per la Filosofia Mitteleuropea.

Menger, Karl 1940 "Topology without Points", *Rice Institute Pamphlets*, 27, 80-107.

Petitot, J. and Smith, B. 1990 "New Foundations for Qualitative Physics", J. E. Tiles, G. T. McKee and C. G. Dean, eds., *Evolving Knowledge in Natural Science and Artificial Intelligence*, London: Pitman Publishing, 231-249.

Petitot, J. and Smith, B. 1993 "Physics and the Phenomenal World", in R. Poli and P. M. Simons, eds., *Formal Ontology*, Dordrecht: Kluwer.

Simons, P. M. 1987 *Parts. A Study in Ontology*, Oxford: Clarendon Press.

Smith, B. 1991 "Relevance, Relatedness and Restricted Set Theory", in G. Schurz and G. J. W. Dorn, eds., *Advances in Scientific Philosophy. Essays in Honour of Paul Weingartner*, Amsterdam/Atlanta: Rodopi, 45-56.

Smith, B. 1992 "Characteristica Universalis", in K. Mulligan, ed., *Language, Truth and Ontology* (Philosophical Studies Series), Dordrecht/Boston/London: Kluwer, 50-81.

Smith, B. ed. 1982 *Parts and Moments. Studies in Logic and Formal Ontology*, Munich: Philosophia. Tarski, A. 1956 "Foundations of the Geometry of Solids", in Tarski, *Logic, Semantics, Metamathematics*, Oxford: Clarendon Press, 24-29.

Tiles, J. E. 1981 *Things That Happen*, Aberdeen: Aberdeen University Press.

Whitehead, Alfred North 1906 "On Mathematical Concepts of the Material World", *Philosophical Transactions of the Royal Society of London*, series A, V. 205, 465-525.

Whitehead, Alfred North 1929 *Process and Reality*, New York: Macmillan.