Systems of granular partitions

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Abstract

We represent granular partitions as triples consisting of a rooted tree structure as first component, a domain satisfying the axioms of General Extensional Mereology as second component, and a (projection) mapping of the first into the second as a third component. We show that granular partitions allow us to model important aspects of the granular and selective character of common sense. We define equivalence and ordering relations among those granular partitions and we show that granular partitions form frame Structures in the sense of [Res00]. This will provide the foundations for a logic which has a semantic that is based on granular partitions.

1 Introduction

Human beings have a variety of ways of dividing up, classifying, mapping, sorting and listing the objects in reality. The theory of granular partitions presented by Bittner and Smith in [BS02] seeks to provide a general and unified basis for understanding such phenomena in formal terms. Its aim is to contribute to an understanding of the granular and selective character of human common sense. Related work in this area includes [Hob85, BWJ98, Ste, Ste00, Don01, Bit02].

The theory of granular partitions has two parts. The first is a theory of classification (Theory A). It describes the tree structures of these classificatory systems. The second is a theory of reference or intentionality (Theory B). It provides an account of how those tree-structures relate to objects in reality.

Consider, for example, the left part of Figure 1. At the left hand side we have a tree-representation of the (incomplete) subdivision of the category *food* into subcategories *fruit* and *vegetables*. Theory A governs how to build nested *cell structures* in such a way that they correspond to the mentioned category trees. In the middle of Figure 1 such a cell structure is represented as Venn-diagram. Theory B governs the way these cell-structures project onto reality indicated by the arrows connecting the middle and the right parts of the Figure.

Bittner and Smith use projection to characterize the relation between the cells in a partition and objects in reality. Briefly, we can think of cells as being projected onto objects in something like the way in which flashlights are projected upon the objects which fall within their purview. This projection corresponds to the way proper names

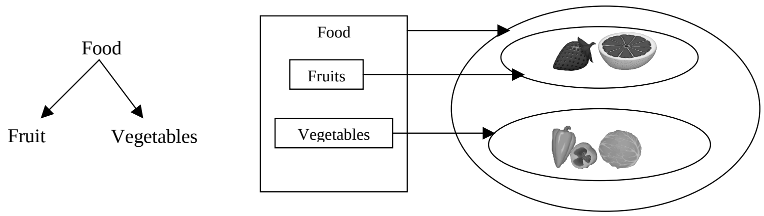

Figure 1: Relationships between cells and objects

project onto or refer to the objects they denote and to the way our acts of perception are related to their objects. (Projection is close to what philosophers call 'intentionality' [Ser83].) Consider 1. Here the cell labeled 'Vegetables' projects this way onto the mereological sum of all vegetables in reality.

Granular partitions are not only at work in the realm of classes of things such as food, vegetables, etc. but also in the realm of individuals. Consider Figure 2. On the left hand side we have the tree representation of certain aspects of the mereological structure of your friend Fred. In the middle we have a corresponding cell structure and at the right hand side we have the target domain – your friend Fred. We assume the obvious 'Head' $\mapsto$ *head*, 'Limbs' $\mapsto$ *left arm + right arm + left leg + right leg* ... projection.

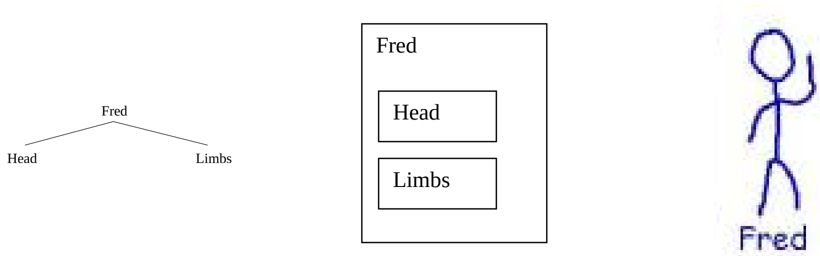

Figure 2: Relationships between cells and objects (2)

All granular partitions both granular and selective: Granularity of projection means that a partition can project onto a whole without projecting onto all of its parts; Selectivity of projection means that a partition does not project onto all objects. Consider Figure 2. The partition is granular since there is a cell projecting onto Fred's head but there are no cells projecting onto parts of Fred's head such as his nose, his ears, etc. and similarly for the other cells which do not have a subcells.

In order to see what selectivity means, consider the cell structure in the middle of Figure 2. Here we have only the subcells 'Head' and 'Limbs'. There is no cell 'Torso' in this cell structure. This may be because this cell tree is a part of a partition which deals only with parts of Fred that 'stick out of the torso'. In this case, the partition selectively projects only on parts which are relevant in the given context.

In their paper [BS02], Bittner and Smith focus on single granular partitions and

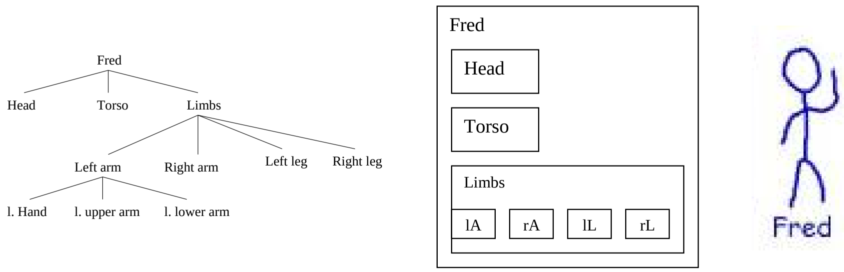

Figure 3: Relationships between cells and objects (3)

their projective relation to reality. In the present paper, we will talk about the relations between granular partitions and define structures on sets of granular partitions. Consider Figures 2 and 3. The granular partitions in both figures project onto Fred, but the partition in Figure 3 includes more detail than the partition in Figure 2. In this paper, we will define a *refinement* relation on partitions according to which the partition in Figure 3 is a refinement of the partition in Figure 2.

To better understand these kinds of relations among granular partitions, we will introduce a class of structures called *labeled granular partitions* and define ordering in these structures. We will show that classes of these structures form *frame structures* in the sense of [Res00].

2 Labeled granular partitions

In this section, we expand the theory of granular partitions presented in [BS02] in two ways. First, we make the mereological structure of the partition cells and the target domain explicit. Second, we add a mechanism for labeling the cells in a partition. We achieve both of these aims by introducing *labeled granular partitions* which can be seen as a more sophisticated version of the *granular partitions* of [BS02].

We begin by presenting the two mereological systems that are needed for the definition of labeled granular partitions.

2.1 Systems of mereology

The primitive relation of mereology is the part-of relation. This binary relation is reflexive, antisymmetric, and transitive, i.e., a partial ordering relation. As pointed out by authors such as [WCH87, GP95, AFG96], there are different kinds of parthood relations which can be further classified by additional axioms. In this paper two kinds parthood relations are of relevance:

1. The parthood relation characterized by the axiomatic system of general extensional mereology (GEM) [Sim87, CV99]. We will use the symbol $\leq$ for this relation. We call the members of the domains of GEM models *objects*.[1]

[1] 'Object' here is used in a very wide sense, to include also scattered mereological sums. Thus a partition

2. The parthood relation characterized by what we call rooted tree mereology (RTM). We will use the symbol $\subseteq$ for this relation.[2] We call the members of the domains of RTM models *cells*.

To give the additional axioms for GEM and RTM, we need to introduce an additional mereological relation. We say that x and y *overlap* if and only if there is some z that is a part of both x and y. We will use the same symbol O for overlap in both GEM and RTM. The formal definitions of the overlap relation in GEM and RTM can be stated as follows.

$$\text{DO-GEM} \quad xOy \equiv \exists z(z \leq x \wedge z \leq y)$$
$$\text{DO-RTM} \quad xOy \equiv \exists z(z \subseteq x \wedge z \subseteq y)$$

In general extensional mereology (GEM) there are two additional axioms besides those requiring $\leq$ to be a partial ordering [Sim87]: the axiom of extensionality and the summation axiom. The axiom of extensionallity tells us that if every object that overlaps x also overlaps y, then x is a part of y:

$$\text{AE-GEM} \quad \forall z(zOx \rightarrow zOy) \rightarrow x \leq y$$

Note that it follows from AE-GEM and the anti-symmetry of $\leq$ that O is extensional in GEM.

$$\text{TE-GEM} \quad \forall z(zOx \leftrightarrow zOy) \rightarrow x = y$$

The summation axiom tells us that, for any model of GEM, if any member of the domain satisfies the formula ϕ then the sum of all members of the domain that satisfy ϕ is also included in the domain.[3]

$$\text{AS-GEM} \quad \exists x \phi x \rightarrow \exists z \forall y(yOz \leftrightarrow \exists x(\phi x \wedge yOx))$$

Structures which are models of rooted tree mereology (RTM) form rooted trees. This is ensured by the axioms below, which are added to the axioms requiring $\subseteq$ to be a partial ordering.

We use the following definition in the axioms.

$$\text{DI-RTM} \quad x \overline{\subseteq} y \equiv x \subseteq y \wedge \forall z(x \subseteq z \subseteq y \rightarrow z = x \vee z = y)$$

When x $\overline{\subseteq}$ y, we say that x is an *immediate subcell* of y.

We now give the following axioms for the partial ordering $\subseteq$:

$$\text{ARoot-RTM} \quad \exists z \forall x x \subseteq z$$

of the animal kingdom might involve a cell labeled *cat*, which projects onto the single object which is the mereological sum of all live cats.

[2]It is important not to confuse this interpretation of $\subseteq$ with the (more standard) subset relation in set theory.

[3]In the axiom, ϕ can be any first-order formula in which x occurs free.

ARoot-RTM requires that each model, Z, of RTM have a maximal cell. It follows from the anti-symmetry of $\subseteq$ that this maximal cell is unique. We will let *root(Z)* stand for the unique maximal cell of the cell tree Z.

AChain-RTM	each cell $z \in Z$ there is a finite chain $z \overline{\subseteq} z_1 \overline{\subseteq} \ldots z_n \overline{\subseteq}$root$(Z)$ of immediate subcells connecting z to root(Z);
AO-RTM	$xOy \rightarrow x \subseteq y \vee y \subseteq x$

AO-RTM restricts overlap to cells that stand in the subcell relation. Thus, there are no instances of proper overlap in RTM models. Notice that it follows from AO-RTM and the anti-symmetry of $\subseteq$ that the graph induced by $\subseteq$ contains no circles, i.e. is a tree.

Finally, because the cell trees in partitions are cognitive artifacts, we add the following axiom.

AFin-RTM There are only finitely many cells in any model of RTM.

Note that models of RTM need not satisfy either the axiom of extensionality or the summation axiom. The axiom of extensionality will fail in models that include a cell, x, which has exactly one immediate proper subcell, y. In this case, x and y will be distinct even though they overlap exactly the same cells. We allow these kinds of models because we want our cell trees to be able to represent the selectivity of human cognition. For example, in a partition representing the parts of a particular yacht, Maude, the cell representing the whole boat, Maude, may have only one proper subcell representing, Maude's engine, because in a particular context we may only be interested in Maude's engine parts.

We do not require that the summation axiom hold, because the classes in taxonomies do not in general combine to form additional classes in the taxonomy. For example, in the standard taxonomy of the animal kingdom, there is no species that is the mereological sum of *rabbits* and *jellyfish*.

2.2 Granular partitions

We introduce the notation $\mathcal{GEM}$ and $\mathcal{RTM}$ to denote the classes of structures satisfying GEM and RTM.

We now define *granular partitions*[4] x as triples of the form

$$(\mathcal{Z}, \Delta, \rho)$$

where $\mathcal{Z} \in \mathcal{RTM}$ is called the *cell tree* of the partition, $\Delta \in \mathcal{GEM}$ is called the *target domain* of the partition, and the *projection-mapping* of signature $\rho : \mathcal{Z} \rightarrow \Delta$ has the following properties:

(i) ρ is a one-one mapping;

(ii) ρ is *order-preserving* in the sense that if $z_1 \subseteq z_2$ then $\rho(z_1) \leq \rho(z_2)$. This ensures that the tree structure in $\mathcal{Z}$ does not distort the mereological structure in Δ;

[4]The formalization of granular partition in this paper differs slightly from the one given in [].

(iii) ρ is not an empty mapping: $\exists z, x : \rho(z) = x$. It follows that every granular partition has at least one cell in its cell tree and at least one object in its target domain ;

(iv) ρ is a total mapping. This equivalent to requiring that granular partitions do not contain empty cells in the sense of [BS02].

In general the ρ will be not an onto mapping due to the selective and granular character of granular partitions.

The formalization of granular partitions presented in this paper corresponds to the one in [BS02] in the following sense that granular partitions $(\mathcal{Z}, \Delta, \phi)$ satisfy the axioms MA1–4 and MB1–6 in [BS02].

2.3 Labeling

Consider the tree structures in Figure 2 and the way the corresponding cell trees project onto the individual *Fred*. The *labels* on the nodes of the tree and the cells are an important aspect of the representations of Fred's parts. This is because without the labeling the cell tree could as well be projected at anything that is a whole with at least two parts (e.g., onto you and your left and right leg).

Since a cell tree has finitely many cells, it is always possible to assign strings of some alphabet α to the cells of a given partition. We now assume that for each partition there is a first-order language $L(\alpha)$ over the alphabet α with a formal semantics of the usual sort where the expressions in $L(\alpha)$ are interpreted over the target domain Δ.

Let $(\mathcal{Z}, \Delta, \rho)$ be a granular partition. A labeling for this partition is a mapping ϕ of signature $\phi : \alpha \to \mathcal{Z}$ assigning strings of the alphabet α to cells in $\mathcal{Z}$. The labeling mappings ϕ in general will be partial since finite partitions do not exhaust all strings of the underlying alphabet. A *labeled granular partition* then is a quintuple of the form

$$(\mathcal{Z}, \Delta, \rho, \alpha, \phi)$$

such that the labeling function ϕ has the following properties:

1. ϕ is one-one, i.e., each cell in the tree $\mathcal{Z}$ has a unique label and

2. ϕ is onto, i.e., every cell in $\mathcal{Z}$ has a label.

3. In order to ensure that labels of cells denote entities in the domain Δ, we demand that if $\alpha_i \in \alpha$ is a label for a cell, (i.e., $\exists z : \phi(\alpha_1) = z$), then there is an entity in Δ which α_i denotes (i.e., $\Delta \models (\exists x : x = \alpha_i)$).

Consider the left part of Figure 4. The corresponding labeled granular partition $(\mathcal{Z}, \Delta, \rho, \alpha, \phi)$ has projection and labeling mappings ρ and ϕ such that the following holds:

$$\rho = \phi(\text{`mammals'}) \mapsto \textit{mammals}, \phi(\text{`cats'}) \mapsto \textit{cats}, \phi(\text{`dogs'}) \mapsto \textit{dogs}. \qquad (1)$$

Here ϕ(‘mammals’) stands for “the cell labeled ‘mammals’ ” and *mammals* refers to the targeted portions of reality (in this case, the mereological sum of all mammals).

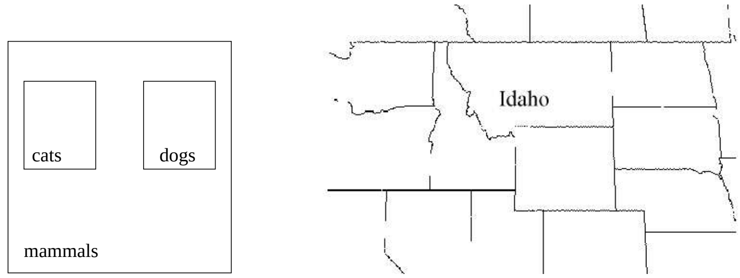

Figure 4: (left) venn-diagram representation of a granular partition; (right) a misslabeled granular partition.

Consider the right part of Figure 4. Here we have a 'missprojection' or 'misslabeling' of the form $\rho((\phi(\text{'Idaho'})) = \textit{Montana}$ which means that the cell labeled 'Idaho' projects onto the piece of land which is usually referred to as Montana. Intuitively this means that the labeling of this partition is in a certain way incompatible with the way the fast majority of other partitions which target the same domain are labeled. In particular, it is incompatible with the way the federal government of the United States labels their maps (which are special kinds of partitions [BS01]). In this particular case, the labeling is not only unusual in the sense that it is incompatible with most other labelings. It is also wrong in the sense that it is true according to the political conventions which make this piece of land a state, it *is* the Federal State Montana.

In order to avoid this kind of 'missprojection' or 'misslabeling' we also demand:

4. if a cell is labeled with the string α_i, then the entity denoted by α_i is identical to the entity projected onto by z, i.e., $z = \phi(\alpha_i) \rightarrow \Delta \models \alpha_i = \rho(z))$.

We will discuss the issues of truth and compatibility of labeling in more detail below.

In the remainder of this paper we will sometimes use the notation 'mammals' instead of $\phi(\text{'mammals'})$ to formally express a sentence-fragment like "the cell labeled 'mammals' ".

3 Relations between granular partitions

3.1 Refinement and extension relations

Consider the granular partitions $(\mathcal{Z}_1, \Delta_1, \rho_1, \alpha_1, \phi_1)$ and $(\mathcal{Z}_2, \Delta_2, \rho_2, \alpha_2, \phi_2)$ in Figures 2 and 3. One can see that the corresponding partitions stand in a kind of refinement relation to each other. We will use the symbol $\preceq$ to refer to this relation and write $(\mathcal{Z}_1, \Delta_1, \rho_1, \alpha_1, \phi_1) \preceq (\mathcal{Z}_2, \Delta_2, \rho_2, \alpha_2, \phi_2)$ to express the fact that the granular partition $(\mathcal{Z}_1, \Delta_1, \rho_1, \alpha_1, \phi_1)$ is a refined by the granular partition $(\mathcal{Z}_2, \Delta_2, \rho_2, \alpha_2, \phi_2)$.

We give a formal account of the relation $\preceq$ as follows. Assume a set of labeled granular partitions P with $(\mathcal{Z}_1, \Delta_1, \rho_1, \alpha_1, \phi_1), (\mathcal{Z}_2, \Delta_2, \rho_2, \alpha_2, \phi_2) \in P$. We then say that $(\mathcal{Z}_1, \Delta_1, \rho_1, \alpha_1, \phi_1) \preceq (\mathcal{Z}_2, \Delta_2, \rho_2, \alpha_2, \phi_2)$ if and only if

$$
\begin{tikzcd}
\alpha \arrow[d, "\phi_1"'] \arrow[dr, "\phi_2"] & \\
Z_1 \arrow[r, "f"] \arrow[d, "\rho_1"'] & Z_2 \arrow[dl, "\rho_2"] \\
\Delta &
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha \arrow[d, "\phi_1"'] \arrow[dr, "\phi_2"] \arrow[drr, "\phi_3"] & & \\
Z_1 \arrow[r, "f"] \arrow[d, "\rho_1"'] & Z_2 \arrow[r, "g"] \arrow[dl, "\rho_2"'] & Z_3 \arrow[dll, "\rho_3"] \\
\Delta & &
\end{tikzcd}
$$

Figure 5: Commutative diagram illustration for: (Left) the definition of $\preceq$; and (Right) the proof of transitivity of $\preceq$.

i The target domains are identical, i.e., $\Delta_1 = \Delta_2$

ii The alphabets underlying the labeling are identical, i.e., $\alpha_1 = \alpha_2$

iii there exists a one-one mapping $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ such that

(a) f is *order-preserving*, i.e., if $z_i \subseteq z_j$ then $f(z_i) \subseteq f(z_j)$,

(b) f is *target-preserving*, i.e., $\rho_1(z) = \rho_2(f(z))$, and

(c) f is *label-preserving*, i.e., $\phi_2(\alpha_i) = f(\phi_1(\alpha_i))$

That the target domains Δ_i and the underlying alphabets α_j are identical (i + ii) reflects the fact that we assume one underlying mereologically structured world and that for each world we label cells of partitions that project at certain parts of this world with strings of the same alphabet.

The existence of the mapping ρ with its particular properties (iii) ensures that we can map cells in $\mathcal{Z}_1$ to cells in $\mathcal{Z}_2$ in such a way that: (a) If two cells in $z_i, z_j \in \mathcal{Z}_1$ are subcells of each other then so are their counterparts in $f(z_1), f(z_2) \in \mathcal{Z}_2$; (b) The target $\rho_1(z)$ of the cell $z \in \mathcal{Z}_1$ is identical to the target $\rho_2(f(z))$ of its counterpart $f(z) \in \mathcal{Z}_2$; and (c) $z \in \mathcal{Z}_1$ and $f(z) \in \mathcal{Z}_2$ have the same labels. In other words we demand that there exists an order-, label, and target-preserving mapping f such that the left diagram in Figure 5 commutes.

We can show that the relation $\preceq$ is reflexive (ref) and transitive (tr):

(ref) We have $(\mathcal{Z}, \rho, \phi) \preceq (\mathcal{Z}, \rho, \phi)$ since the identity map of a cell tree onto itself, defined by $z = id(z)$ is always order-,label-, and target-preserving.

(tr) For transitivity we have to show that if $f : Z_1 \to Z_2$ and $g : Z_2 \to Z_3$ are order-,label- , and target-preserving then so is their composition $g \circ s : Z_1 \to Z3$. That this is the case can be seen in the right diagram in Figure 5.

In the remainder of this section we consider partitions which stand in the $\preceq$ relation. We will simplify the notion $(\mathcal{Z}, \Delta, \rho, \alpha, \phi)$ for more convenient writing by using $(\mathcal{Z}, \rho, \phi)$ in order to refer to a labeled granular partitions. We can do this because the target domains and alphabets of those partitions are identical. We now continue by considering two special cases of refinement: proper refinement and extension.

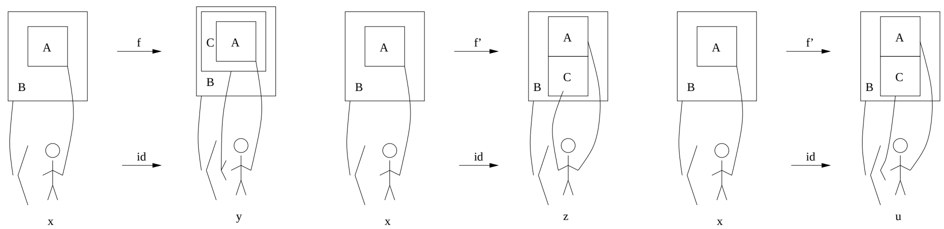

Figure 6: Examples of partitions between the relation $\preceq$ holds.

Proper refinement. Consider the left part of Figure 6. We have a partition $(\mathcal{Z}_x, \rho_x, \phi_x)$ with cells labeled 'A' and 'B' and 'A' $\subseteq$ 'B' and with 'A' projecting onto your friend Freds' right arm, i.e., 'A' $\mapsto$ *Freds' right arm*, and the cell labeled 'B' projecting onto Freds whole body 'B' $\mapsto$ *Freds' body*. (In Figure 6 we use the stretched bracket $<$ to indicate that 'C' targets Fred's whole body.) We also have a partition $(\mathcal{Z}_y, \rho_y, \phi_y)$ with 'A' $\subseteq$ 'C' $\subseteq$ 'B' with 'A' and 'B' as above and with 'C' projecting onto Fred's upper body, i.e., 'C' $\mapsto$ *Freds' upper body*. (In the figure we use the small bracket $<$ to indicate that 'A' targets Fred's upper body.) It is easy to see that the induced mapping $f : \mathcal{Z}_x \to \mathcal{Z}_y$ is order-, identity-, and label-preserving. Thus $(\mathcal{Z}_x, \rho_x, \phi_x) \preceq (\mathcal{Z}_y, \rho_y, \phi_y)$.

The situation in the middle part of Figure 6 is similar. We have $(\mathcal{Z}_x, \rho_x, \phi_x)$ as before. However we have a refinement $(\mathcal{Z}_z, \rho_z, \phi_z)$ in which the cell labeled 'C' is not a supercell of 'A' and in which 'C' projects onto Freds' left arm, i.e., 'C' $\mapsto$ *Freds' left arm*. Again, the induced mapping $g : \mathcal{Z}_x \to \mathcal{Z}_z$ is order-, identity-, and label-preserving. Thus $(\mathcal{Z}_x, \rho_x, \phi_x) \preceq (\mathcal{Z}_z, \rho_z, \phi_z)$ and hence $(\mathcal{Z}_x, \rho_x, \phi_x) \preceq (\mathcal{Z}_z, \rho_z, \phi_z)$.

In the right part of Figure 6, we have a refinement of $(\mathcal{Z}_x, \rho_x, \phi_x)$ by $(\mathcal{Z}_u, \rho_u, \phi_u)$ similar to the refinement in the left part of the figure. The refinement partition $(\mathcal{Z}_y, \rho_y, \phi_y)$ and $(\mathcal{Z}_u, \rho_u, \phi_u)$ recognize the same parts of Fred: Fred as a whole, Freds upper body, and Freds right arm. They differ however in the following respect: The partition $(\mathcal{Z}_y, \rho_y, \phi_y)$ recognizes the fact that Freds right arm is a part of Freds upper body. This aspect of mereological ordering is traced over in the partition $(\mathcal{Z}_u, \rho_u, \phi_u)$.

Note that not only is $(\mathcal{Z}_y, \rho_z, \phi_y)$ is a refinement of $(\mathcal{Z}_x, \rho_x, \phi_x)$. It is also a refinement of $(\mathcal{Z}_u, \rho_u, \phi_u)$. To see this consider the mapping $h : \mathcal{Z}_u \to \mathcal{Z}_y$ mapping cells in $\mathcal{Z}_u$ to cells with matching labels in $\mathcal{Z}_y$. Clearly, h is order-, identity-, and label-preserving, hence $(\mathcal{Z}_u, \rho_u, \phi_u) \preceq (\mathcal{Z}_y, \rho_z, \phi_y)$. On the other hand the partitions $(\mathcal{Z}_y, \rho_y, \phi_y)$ and $(\mathcal{Z}_z, \rho_z, \phi_y)$ are not comparable with respect to $\preceq$ since the cell labeled 'C' in Z_y and the cell labeled 'C' in Z_z target different parts of Freds body. Therefore no commutative diagram like the one in the left of Figure 5 can be constructed for the two partitions.

All of the above refinement relations in Figure 6 are examples of what we call *proper refinement*. In proper refinement the object targeted by the root cell – the cell 'B' in Figure 6 – remains the same. A proper refinement can target additional objects as long as these objects are parts of objects targeted by the original partition (e.g., $x \preceq y$ in the figure). Or, a proper refinement may target the same set of objects but include more information about mereological relations between objects (e.g., $u \preceq y$ in the figure).

Extension. As an example of another way of how a granular partition can be refined consider a granular partition $(\mathcal{Z}_1, \rho_1, \phi_1)$ which recognizes the Federal States of the US and let $(\mathcal{Z}_2, \rho_2, \phi_2)$ represent a granular partition which recognizes the Federal States of the US as well as the states of the European Community together with a root cell 'The United States and the States of the EU'. It is easy to see that we have $(\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_2, \rho_2, \phi_2)$.

This is an example of what we will call *extensions*. When one partition is an extension of another, then the target of the original root cell is always a proper part of the extension's root cell.

Assume $(\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_2, \rho_2, \phi_2)$ and consider the corresponding commutative diagram in the left part of Figure 5. As sketched above, we can further analyze the two different uses of refinement by considering the projection of those cells in $\mathcal{Z}_2$ which are not targeted by the mapping f. Intuitively, in the case of proper refinement those cells project onto objects in Δ_2 which are parts of objects in the image of ρ_1. In the case of extension, those cells project onto objects in Δ_2 which are not parts of objects in the image of ρ_1. Formally we define now define the binary relations *RP* (x is-properly-refined-by y) and EP (x is-properly-extended-by y) which both are subrelations of $\preceq$ as follows:

$$
\begin{aligned}
RP((\mathcal{Z}_1, \rho_1, \phi_1), (\mathcal{Z}_2, \rho_2, \phi_2)) \equiv\quad & (\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_2, \rho_2, \phi_2) \text{ and} \\
& \forall z_2 \in \mathcal{Z}_2(\exists z_1 \in \mathcal{Z}_1(\rho_2(z_2) \leq \rho_1(z_1))). \\
EP((\mathcal{Z}_1, \rho_1, \phi_1), (\mathcal{Z}_2, \rho_2, \phi_2)) \equiv\quad & (\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_2, \rho_2, \phi_2) \text{ and} \\
& \neg RP((\mathcal{Z}_1, \rho_1, \phi_1), (\mathcal{Z}_2, \rho_2, \phi_2)).
\end{aligned}
$$

3.2 Equivalence and ordering

We now continue by defining an equivalence relation on granular partitions $(\mathcal{Z}_1, \rho_1, \phi_1)$ and $(\mathcal{Z}_2, \rho_2, \phi_2)$ by saying that two partitions are equivalent if and only if both stand in the relation $\preceq$ to each other, i.e.,

$$(\mathcal{Z}_1, \rho_1, \phi_1) \sim (\mathcal{Z}_2, \rho_2, \phi_2) \equiv (\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_2, \rho_2, \phi_2) \ \wedge \ (\mathcal{Z}_2, \rho_2, \phi_2) \preceq (\mathcal{Z}_1, \rho_1, \phi_1).$$

The relation $\sim$ is an equivalence relation, i.e., reflexive, symmetric, and transitive. The reflexivity of $\sim$ follows immediately from the reflexivity of $\preceq$.

The symmetry of $\sim$ follows from the symmetric character of its definition. To see the transitivity of $\sim$ assume $(\mathcal{Z}_1, \rho_1, \phi_1) \sim (\mathcal{Z}_2, \rho_2, \phi_2)$ and $(\mathcal{Z}_2, \rho_2, \phi_2) \sim (\mathcal{Z}_3, \rho_3, \phi_3)$. Therefore we have $(\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_2, \rho_2, \phi_2)$ and $(\mathcal{Z}_2, \rho_2, \phi_2) \preceq (\mathcal{Z}_1, \rho_1, \phi_1)$ and similarly $(\mathcal{Z}_2, \rho_2, \phi_2) \preceq (\mathcal{Z}_3, \rho_3, \phi_3)$ and $(\mathcal{Z}_3, \rho_3, \phi_3) \preceq (\mathcal{Z}_2, \rho_2, \phi_2)$. From the transitivity of $\preceq$ it follows that we have $(\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_3, \rho_3, \phi_3)$ and $(\mathcal{Z}_3, \rho_3, \phi_3) \preceq (\mathcal{Z}_1, \rho_1, \phi_1)$ and hence $(\mathcal{Z}_1, \rho_1, \phi_1) \sim (\mathcal{Z}_3, \rho_3, \phi_3)$.

The corresponding set of equivalence classes is defined as

$$[(\mathcal{Z}, \rho, \phi)] = \{(\mathcal{Z}_1, \rho_1, \phi_1) \mid (\mathcal{Z}_1, \rho_1, \phi_1) \sim (\mathcal{Z}, \rho, \phi)\}$$

The elements of $[(\mathcal{Z}, \rho, \phi)]$ are distinct labeled granular partitions with cell trees that have identical structure and whose cells have identical labels and target the same objects in reality. Examples of such an equivalence classes are: (a) Consider several copies of this paper. Then all instances of the the labeled granular partition shown in Figure 2 are members of the equivalence class [the-partition-in-this-figure-2] (the same of course holds for all other partitions in this paper), (b) The set of all current maps of the federal states of the United States, and (c) The set periodic tables of elements in the different text books of chemistry.

Consider equivalence classes $[(\mathcal{Z}_1, \rho_1, \phi_1)]$ and $[(\mathcal{Z}_2, \rho_2, \phi_2)]$. The relation $\preceq$ now induces a partial ordering $\ll$ as follows:

$$[(\mathcal{Z}_1, \rho_1, \phi_1)] \ll [(\mathcal{Z}_2, \rho_2, \phi_2)] \equiv (\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_2, \rho_2, \phi_2) \tag{2}$$

To show that $\ll$ is well defined suppose $(\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_2, \rho_2, \phi_2)$ and $(\mathcal{Z}_x, \rho_x, \phi_x) \in [\mathcal{Z}_1, \rho_1, \phi_1]$ and $(\mathcal{Z}_y, \rho_y, \phi_y) \in [\mathcal{Z}_2, \rho_2, \phi_2]$. Then $(\mathcal{Z}_x, \rho_x, \phi_x) \preceq (\mathcal{Z}_1, \rho_1, \phi_1)$ and $(\mathcal{Z}_2, \rho_2, \phi_2) \preceq (\mathcal{Z}_y, \rho_y, \phi_y)$. By transitivity of $\preceq$, $(\mathcal{Z}_x, \rho_x, \phi_x) \preceq (\mathcal{Z}_y, \rho_y, \phi_y)$.

The relation $\ll$ is a partial ordering. The reflexivity and transitivity of $\ll$ immediately follow from the reflexivity and transitivity of $\preceq$. It remains to show that $\ll$ is antisymmetric, i.e., that if $[(\mathcal{Z}_1, \rho_1, \phi_1)] \ll [(\mathcal{Z}_2, \rho_2, \phi_2)]$ and $[(\mathcal{Z}_2, \rho_2, \phi_2)] \ll [(\mathcal{Z}_1, \rho_1, \phi_1)]$ then $[(\mathcal{Z}_1, \rho_1, \phi_1)] = [(\mathcal{Z}_2, \rho_2, \phi_2)]$. If $[(\mathcal{Z}_1, \rho_1, \phi_1)] \ll [(\mathcal{Z}_2, \rho_2, \phi_2)]$ and $[(\mathcal{Z}_2, \rho_2, \phi_2)] \ll [(\mathcal{Z}_1, \rho_1, \phi_1)]$ holds then so does $(\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_2, \rho_2, \phi_2)$ and $(\mathcal{Z}_2, \rho_2, \phi_2) \preceq (\mathcal{Z}_1, \rho_1, \phi_1)$. Hence we have $(\mathcal{Z}_1, \rho_1, \phi_1) \sim (\mathcal{Z}_2, \rho_2, \phi_2)$ and thus $[(\mathcal{Z}_1, \rho_1, \phi_1)] = [(\mathcal{Z}_2, \rho_2, \phi_2)]$.

In the remainder we will write $[\mathcal{Z}, \rho, \phi]$ instead of $[(\mathcal{Z}, \rho, \phi)]$ to simplify the notation.

3.3 Composition of granular partitions

Consider Figure 7. We have a granular partition $(\mathcal{Z}_{FL}, \rho_{FL}, \phi_{FL})$ whose cells project onto the left part of Freds body (FL) and we have a granular partition $(\mathcal{Z}_{FR}, \rho_{FR}, \phi_{FR})$ whose cells project onto the right part of Freds body (FR). We now want to define a composition operation *comp*, which when applied to $(\mathcal{Z}_{FL}, \rho_{FL}, \phi_{FR})$ and $(\mathcal{Z}_{FR}, \rho_{FR}, \phi_{FR})$, yields the partition $(\mathcal{Z}_{WF}, \rho_{WF}, \phi_{WF})$ whose cells target the left as well as the right part of Freds body, i.e., whole Fred (WF), as indicated in the figure.

We define a ternary relation *comp* as follows:

$$\begin{aligned} comp\,(\mathcal{Z}_x, \rho_x, \phi_x)(\mathcal{Z}_y, \rho_y, \phi_y)(\mathcal{Z}_z, \rho_z, \phi_z) \equiv \quad & (\mathcal{Z}_x, \rho_x, \phi_x) \preceq (\mathcal{Z}_z, \rho_z, \phi_z) \wedge \\ & (\mathcal{Z}_y, \rho_y, \phi_y) \preceq (\mathcal{Z}_z, \rho_z, \phi_z). \end{aligned}$$

This definition if well-defined since the definition of $\preceq$ is well defined. Since $\preceq$ is reflexive we have *comp* $(\mathcal{Z}_x, \rho_x, \phi_x)(\mathcal{Z}_x, \rho_x, \phi_x)(\mathcal{Z}_x, \rho_x, \phi_x)$ and from the definition of

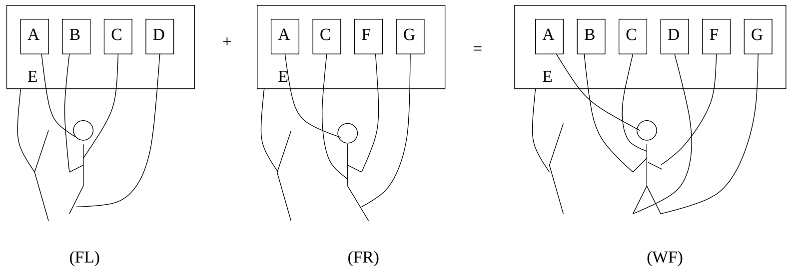

Figure 7: Composition of partitions

comp it follows immediately that it is commutative in the first two arguments, i.e., if *comp* $(\mathcal{Z}_x, \rho_x, \phi_x)(\mathcal{Z}_y, \rho_y, \phi_y)(\mathcal{Z}_z, \rho_z, \phi_z)$ then *comp* $(\mathcal{Z}_y, \rho_y, \phi_y)(\mathcal{Z}_x, \rho_x, \phi_x)(\mathcal{Z}_z, \rho_z, \phi_z)$.

Given this composition relation one can ask under which circumstances composition is an operation and under which circumstances this operation then is associative. This discussion however is omitted here due to space limitations.

Consider the equivalence classes $[\mathcal{Z}_1, \rho_1, \phi_1]$, $[\mathcal{Z}_2, \rho_2, \phi_2]$, and $[\mathcal{Z}_3, \rho_3, \phi_3]$ the composition relation *comp* now induces a corresponding relation $\oplus$ as follows

$$\oplus[\mathcal{Z}_1, \rho_1, \phi_1][\mathcal{Z}_2, \rho_2, \phi_2][\mathcal{Z}_3, \rho_3, \phi_3] \equiv \textit{comp}\ (\mathcal{Z}_1, \rho_1, \phi_1)(\mathcal{Z}_2, \rho_2, \phi_2)(\mathcal{Z}_3, \rho_3, \phi_3).$$

To see that $\oplus$ is well-defined assume $\oplus[\mathcal{Z}_1, \rho_1, \phi_1][\mathcal{Z}_2, \rho_2, \phi_2][\mathcal{Z}_3, \rho_3, \phi_3]$. From Definition 2 it follows that we have $[\mathcal{Z}_1, \rho_1, \phi_1] \ll [\mathcal{Z}_3, \rho_3, \phi_3]$ and $[\mathcal{Z}_2, \rho_2, \phi_2] \ll [\mathcal{Z}_3, \rho_3, \phi_3]$. Now chose $(\mathcal{Z}_x, \rho_x, \phi_x) \in [\mathcal{Z}_1, \rho_1, \phi_1]$, $(\mathcal{Z}_y, \rho_y, \phi_y) \in [\mathcal{Z}_2, \rho_2, \phi_2]$, and $(\mathcal{Z}_z, \rho_z, \phi_z) \in [\mathcal{Z}_3, \rho_3, \phi_3]$. Therefore we have $(\mathcal{Z}_x, \rho_x, \phi_x) \preceq (\mathcal{Z}_z, \rho_z, \phi_z)$ and $(\mathcal{Z}_y, \rho_y, \phi_y) \preceq (\mathcal{Z}_z, \rho_z, \phi_z)$ and hence *comp* $(\mathcal{Z}_x, \rho_x, \phi_x)(\mathcal{Z}_z, \rho_z, \phi_z)(\mathcal{Z}_z, \rho_z, \phi_z)$.

4 Partition frames

Let P be the set of labeled granular partitions which target the domain Δ and which cells are labeled with strings of the alphabet α. Now let $\mathcal{P} = \{[x]_\sim \mid x \in P\}$ be the set of equivalence classes which are induced by the equivalence relation $\sim$. We call $(\mathcal{P}, \ll, \oplus)$ a *partition frame*.

We will show that $\ll$ has the properties Restall [Res00] calls plumb positive binary accessibility relation and plumb negative binary accessibility relation and that $\oplus$ is a plumb ternary accessibility relation in Restall's sense, and that $\mathcal{P}$ is what Restall calls a truthset.

Let X be a set and let $\leq$ be a partial order on X. A relation S is a *plump* binary *positive* accessibility relation if and only if

$$\forall x, x', y, y' \in X: \text{ (if } xSy \text{ and } x' \leq x \text{ and } y \leq y') \text{ then } x'Sy' \ .$$

A relation C is a *plumb negative* binary accessibility relation if and only if

$$\forall x, x', y, y' \in X: \text{ (if } xCy \text{ and } x' \leq x \text{ and } y' \leq y) \text{ then } x'Cy'.$$

We now take the underlying partially ordered set to be the set of equivalence classes of labeled granular partitions $(\mathcal{P}, \ll)$. That $\ll$ has the properties of a plump positive binary accessibility relation and a plump negative binary accessibility relation follows immediately the definition of $\ll$ and from the transitivity of $\preceq$.

Let X be defined as above. A relation R is a plumb ternary accessibility relation if and only if

$$\forall x, y, x', y', z, z' \text{ if } Rxyz \text{ and } x' \preceq x \text{ and } y' \preceq y \text{ and } z \preceq z' \textit{ then } Rx'y'z'.$$

Again we take $(\mathcal{P}, \ll)$ as the underlying partially ordered set and we take $\oplus$ to be the plumb ternary accessibility relation R. To see that $\oplus$ indeed is a plumb ternary accessibility relation consider the following. From the well formedness of the definition of $\ll$ it follows that it is sufficient to consider arbitrary $(\mathcal{Z}_1, \rho_1, \phi_1) \in [\mathcal{Z}_1, \rho_1, \phi_1]$, $(\mathcal{Z}_2, \rho_2, \phi_2) \in [\mathcal{Z}_2, \rho_2, \phi_2]$, etc. and to demonstrate the property for *comp*. To see that *comp* has this property consider the following: If *comp* $(\mathcal{Z}_1, \rho_1, \phi_1)(\mathcal{Z}_2, \rho_2, \phi_2)(\mathcal{Z}_3, \rho_3, \phi_3)$ holds then we have $(\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_3, \rho_3, \phi_3)$ and $(\mathcal{Z}_2, \rho_2, \phi_2) \preceq (\mathcal{Z}_3, \rho_3, \phi_3)$ by the definition of *comp*. Therefore we have $(\mathcal{Z}_x, \rho_x, \phi_x) \preceq (\mathcal{Z}_1, \rho_1, \phi_1) \preceq (\mathcal{Z}_3, \rho_3, \phi_3) \preceq (\mathcal{Z}_z, \rho_z, \phi_z)$ and $(\mathcal{Z}_y, \rho_y, \phi_y) \preceq (\mathcal{Z}_2, \rho_2, \phi_2) \preceq (\mathcal{Z}_3, \rho_3, \phi_3) \preceq (\mathcal{Z}_z, \rho_z, \phi_z)$. By transitivity it follows that $(\mathcal{Z}_x, \rho_x, \phi_x) \preceq (\mathcal{Z}_z, \rho_z, \phi_z)$ and $(\mathcal{Z}_y, \rho_y, \phi_y) \preceq (\mathcal{Z}_z, \rho_z, \phi_z)$ and therefore *comp* $(\mathcal{Z}_x, \rho_x, \phi_x)(\mathcal{Z}_y, \rho_y, \phi_y)(\mathcal{Z}_z, \rho_z, \phi_z)$.

Let X be defined as above and R be a ternary accessibility relation. Restall calls a set $T \subseteq X$ a *truthset* if and only if

$$\forall x, y \in X(x \leq y \text{ if and only if } \exists z \in T(Rzxy \text{ and } Rxzy))$$

Again we take $(\mathcal{P}, \ll)$ as the underlying partially ordered set and we take $\oplus$ to be the plumb ternary accessibility relation. That $\mathcal{P}$ is a truthset then follows immediately from the definition of $\oplus$.

5 Relation between partitions and formulas

So far we only considered the relationship of labeling between the strings of some alphabet α and cells in granular partitions. The strings in α hereby where understood implicitly in a twofold manner: (i) as labels of cells, and (ii) as names of entities which are targeted by the cells they label (e.g., Figures 6 and 7). We now consider the partition-theoretic semantics of a prepositional logic $\mathcal{L}(\alpha_\exists)$ over the alphabet $\alpha_\exists$. The alphabet $\alpha_\exists$ is obtained from α by transforming the name $\alpha_i \in \alpha$ into the preposition $(\exists x : x = \alpha_i) \in \alpha_\exists$. We use $p, p_1, q, \ldots$ as names of variables ranging over prepositions of the form '$\exists x : x = \alpha_i$'.

The logic $\mathcal{L}(\alpha_\exists)$ then is obtained as follows: $p, p_1, q, \ldots$ are atomic *formulas* in $\mathcal{L}(\alpha_\exists)$. $A, B, \ldots$ are complex formulas which are defined recursively as follows: If $A, B \in \mathcal{L}(\alpha_\exists)$ then so are $A \wedge B$, $A \vee B$, $A \to B$, $A \circ B$, $\neg A$, $\Box A$, and $\Diamond A$. X, Y, and Z are sets of formulas. If X and $Y \in \mathcal{L}(\alpha_\exists)$ then so are X, Y and $X; Y$ where the punctation marks , and ; refer to different ways of combining sets of formulas in $\mathcal{L}(\alpha_\exists)$.

Let $\mathcal{L}(\alpha_\exists)$ be a partition logic, let $\Pi = [(\mathcal{Z}, \Delta, \rho, \alpha, \phi)]$ be an equivalence class of granular partitions as defined above, and let p be an atomic formula of the form $\exists x(x = \alpha_i)$. We now define:

$$\Pi \Vdash \mathrm{p} \equiv \forall \pi \in \Pi(\exists z \in Z_\pi : \phi_\pi(\alpha_i) = z \wedge \rho_\pi(z) \models \exists x(x = \alpha_i)) \tag{3}$$

$\Pi \Vdash \mathrm{p}$ then is interpreted as 'The partitions $\pi \in \Pi$ support that p holds'. We write $\Pi \nVdash \mathrm{p}$ if and only if $\neg(\Pi \Vdash \mathrm{p})$.

We then can define the interpretation of complex formulas as follows:

$$\begin{array}{lcl} \Pi \Vdash A \wedge B & \equiv & \Pi \Vdash A \text{ and } \Pi \Vdash B \\ \Pi \Vdash A \vee B & \equiv & \Pi \Vdash A \text{ or } \Pi \Vdash B \\ \Pi \Vdash \Box A & \equiv & \forall \Pi_1 (\text{ if } \Pi\ S\ \Pi_1 \text{ then } \Pi_1 \Vdash A) \\ \Pi \Vdash \Diamond A & \equiv & \exists \Pi_1 (\text{ if } \Pi\ S\ \Pi_1 \text{ then } \Pi_1 \Vdash A) \\ \Pi \Vdash -A & \equiv & \forall \Pi_1 (\text{ if } \Pi\ C\ \Pi_1 \text{ then } \pi_1 \nVdash A \\ \Pi \Vdash \neg A & \equiv & \forall \Pi_1 (\text{ if } \Pi_1\ C\ \Pi \text{ then } \pi_1 \nVdash A \end{array}$$

6 Conclusions and future work

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